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Dimensional regularization of the third post-Newtonian dynamics of point particles in harmonic coordinates

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Abstract

Dimensional regularization is used to derive the equations of motion of two point masses in harmonic coordinates. At the third post-Newtonian (3PN) approximation, it is found that the dimensionally regularized equations of motion contain a pole part [proportional to $(d-3)^{-1}$] which diverges as the space dimension d tends to 3. It is proven that the pole part can be renormalized away by introducing suitable shifts of the two world-lines representing the point masses, and that the same shifts renormalize away the pole part of the "bulk" metric tensor $g_{\mu\nu}(x^\lambda)$. The ensuing, finite renormalized equations of motion are then found to belong to the general parametric equations of motion derived by an extended Hadamard regularization method, and to uniquely determine the 3PN ambiguity parameter λ to be: $\lambda = -1987/3080$. This value is fully consistent with the recent determination of the equivalent 3PN "static ambiguity" parameter, $\omega_s = 0$, by a dimensional-regularization derivation of the Hamiltonian in Arnowitt-Deser-Misner coordinates. Our work provides a new, powerful check of the consistency of the dimensional regularization method within the context of the classical gravitational interaction of point particles.

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I. INTRODUCTION

A. Relation to previous works

The problem of motion, one of the cardinal problems of Einstein's gravitation theory, has received continuous attention over the years. The early, classic works of Lorentz-Droste, Eddington-Clark, Einstein-Infeld-Hoffmann, Fock, Papapetrou and others led to a good understanding of the equations of motion of N bodies at the first post-Newtonian (1PN) approximation [1] (see, *e.g.*, [1] for a general review of the problem of motion). In the 1970's, an important series of works by a Japanese group [2, 3, 4] led to a nearly complete control of the problem of motion at the second post-Newtonian (2PN) approximation. Then, in the early 80's, motivated by the observation of secular orbital effects in the Hulse-Taylor binary pulsar PSR1913+16, several groups solved the two-body problem at the 2.5PN level (while completing on the way the derivation of the 2PN dynamics) [5, 6, 7, 8, 9, 10, 11, 12] (for more recent work on the 2.5PN dynamics see [13, 14, 15]).

In the late 90's, motivated by the aim of deriving high-accuracy templates for the data analysis of the upcoming international network of interferometric gravitational-wave detectors, two groups embarked on the derivation of the equations of motion at the third post-Newtonian (3PN) level. One group used the Arnowitt-Deser-Misner (ADM) Hamiltonian approach [16, 17, 18, 19, 20] and worked in a corresponding ADM-type coordinate system. Another group used a direct post-Newtonian iteration of the equations of motion in harmonic coordinates [21, 22, 23, 24, 25, 26]. The end results of these two approaches have been proved to be physically equivalent [20, 25]. However, both approaches, even after exploiting all symmetries and pushing their Hadamard-regularization-based methods to the maximum of their possibilities, left undetermined *one and only one* dimensionless parameter: ω_s in the ADM approach and λ in the harmonic-coordinates one. The unknown parameters in both approaches are related by

$$\lambda = -\frac{3}{11}\omega_s - \frac{1987}{3080}\,, \tag{1.1}$$

as was deduced from the comparison between the invariant energy functions for circular orbits in the two approaches [21], and from two independent proofs of the equivalence between the two formalisms for general orbits [20, 25]. The appearance of one (and only one) unknown parameter in the equations of motion is quite striking; it is related with the choice of the regularization method used to cure the self-field divergencies of point particles. Both lines of works [16, 17, 18, 19, 20] and [21, 22, 23, 24, 25, 26] regularized the self-field divergencies by some version of the Hadamard regularization method. The second line of work defined an extended version of the Hadamard regularization [23, 24], which permitted a self-consistent derivation of the 3PN equations of motion, but its use still allowed for the presence of arbitrary parameters in the final equations. On the other hand, the Hadamard regularization also yielded some arbitrary parameters in the gravitational radiation field of point-particle binaries at the 3PN order, the most important of which being the parameter θ entering the binary's energy flux [27, 28].

Let us notice that the regularization (when dealing with point particles) and the renormalization (needed when dealing either with point particles or with extended bodies [2]) of

[1] As usual the nPN order refers to the terms of order $1/c^{2n}$ in the equations of motion.

[2] In the case of extended compact bodies, the gravitational self-energy (divergent when the radii of the

self-field effects has recurrently plagued the general relativistic problem of motion. Even at the 1PN level, early works often contained incorrect treatments of self-field effects (see, *e.g.*, Section 6.14 of [1] for a review). At the 2PN level, the self-field divergencies are more severe than at the 1PN level. For instance, they caused Ref. [3] to incorrectly evaluate the "static" (*i.e.*, velocity-independent) part of the 2PN two-body Hamiltonian. The first correct and complete evaluation of the 2PN dynamics has been obtained by using the Riesz analytical continuation method [29]. (See [8, 11] for a detailed discussion of the evaluation of the static 2PN two-body Hamiltonian.) In brief, the Riesz analytical continuation method consists of replacing the problematic delta-function stress-energy tensor of a set of point particles $y_a^\mu(s_a)$,

$$T^{\mu\nu}(x) = \sum_a m_a\, c^2 \int ds_a\, \frac{dy_a^\mu}{ds_a}\,\frac{dy_a^\nu}{ds_a}\,[-g(y_a)]^{-1/2}\,\delta^{(4)}(x^\lambda - y_a^\lambda(s_a))\,, \tag{1.2}$$

[where $ds_a^2 = -g_{\mu\nu}(y_a^\lambda)\, dy_a^\mu\, dy_a^\nu$, $g \equiv \det g_{\mu\nu}$] by an auxiliary, smoother source

$$T_A^{\mu\nu}(x) = \sum_a m_a\, c^2 \int ds_a\, \frac{dy_a^\mu}{ds_a}\,\frac{dy_a^\nu}{ds_a}\,[-g(y_a)]^{-1/2}\,Z_A^{(4)}(x^\lambda - y_a^\lambda(s_a))\,. \tag{1.3}$$

[Actually, in the implementation of [8], one works with $\mathcal{T}^{\mu\nu}(x) \equiv |g(x)|\, T^{\mu\nu}(x)$.] In Eq. (1.3) the four-dimensional delta function entering Eq. (1.2) has been replaced by the Lorentzian [3] four-dimensional Riesz kernel $Z_A^{(4)}(x-y)$, which depends on the complex number A. When the real part of A is large enough the source $T_A^{\mu\nu}(x)$ is an ordinary function of x^μ, which is smooth enough to lead to a well-defined iteration of the harmonically relaxed Einstein field equations, involving no divergent integrals linked to the behavior of the integrands when $x^\mu \to y_a^\mu$. One then analytically continues A down to 0, where the kernel $Z_A^{(4)}(x-y)$ tends to $\delta^{(4)}(x-y)$. The important point is that it has been shown [8] that all the integrals appearing in the 2PN equations of motion are meromorphic functions of A which admit a smooth continuation at $A = 0$ (without poles). It was also shown there that the formal construction based on (1.3) does generate, at the 2.5PN level, the metric and equations of motion of N "compact" bodies (*i.e.*, bodies with radii comparable to their Schwarzschild radii).

The Riesz analytic continuation method just sketched works within a normal 4-dimensional space-time (as recalled by the superscript (4) in (1.3)). However, it was mentioned in [30] that the same final result (at the 2.5PN level) is obtained by replacing $Z_A^{(4)}(x-y)$ by $Z_0^{(4-A)}(x-y) \equiv \delta^{(4-A)}(x-y)$, *i.e.*, by formally considering delta-function sources in a space-time of complex dimension $4-A$. In other words, at the 2.5PN level, the Riesz analytic continuation method is equivalent to the *dimensional regularization* method.[4] However, it was also mentioned at the time [8] that the generalization of Riesz analytic continuation beyond the 2.5PN level did not look straightforward because of the appearance of poles, proportional to A^{-1}, at the 3PN level (when using harmonic coordinates).

bodies formally tend to zero) must be renormalized into the definition of the mass.

[3] $Z_A^{(d+1)}$ is the Lorentzian version of the Euclidean kernel $\delta_\alpha^{(d)}$ discussed in Appendix B.

[4] Dimensional regularization was invented as a mean to preserve the *gauge symmetry* of perturbative *quantum* gauge theories [31, 32, 33, 34]. Our basic problem here is to respect the gauge symmetry associated with the *diffeomorphism invariance* of the *classical* general relativistic description of interacting point masses.

Recently, Damour, Jaranowski and Schäfer [35] showed how to use dimensional regularization within the ADM canonical formalism. They found that the reduced Hamiltonian describing the dynamics of two point masses in space-time dimension $D \equiv d+1$ was *finite* (no pole part) as $d \to 3$. They also found that the unique 3PN Hamiltonian defined by the analytic continuation of d towards 3 had two properties: (i) the velocity-dependent terms had the unique structure compatible with global Poincaré invariance,[5] and (ii) the velocity-independent ("static") terms led to an unambiguous determination of the unknown ADM parameter ω_s, namely

$$\omega_s^{\text{dim. reg. ADM}} = 0\,. \tag{1.4}$$

The fact that the dimensionally regularized 3PN ADM Hamiltonian ends up being globally Poincaré invariant is a confirmation of the consistency of dimensional regularization, because this symmetry is not at all manifest within the ADM approach which uses a space-plus-time split from the start. By contrast, the global Poincaré symmetry is manifest in harmonic coordinates, and indeed the 3PN harmonic-coordinates equations of motion derived in [21, 22] were found to be manifestly Poincaré invariant.

B. Method and main results

In the present paper, we shall show how to implement dimensional regularization (henceforth often abbreviated as "dim. reg." or even "dr") in the computation of the equations of motion in harmonic coordinates, *i.e.*, following the same iterative post-Newtonian formalism as in Refs. [13, 21, 22]. Similarly to the ADM calculation of Ref. [35], our strategy will essentially consist of computing the *difference* between the d-dimensional result and the 3-dimensional one [21, 22] corresponding to Hadamard regularization. This difference is computed in the form of a Laurent expansion in $\varepsilon \equiv d-3$, where d denotes the spatial dimension. The main reason for computing the ε-expansion of the difference is that it depends only on the singular behavior of various metric coefficients in the vicinity of the point particles, so that the functions involved in the delicate divergent integrals can all be computed in d dimensions in the form of local expansions in powers of r_1 or r_2 (where $r_a \equiv |\mathbf{x}-\mathbf{y}_a|$; $\mathbf{y}_a$, $a=1,2$, denoting the locations of the two point masses). Dimensional regularization as we use it here can then be seen as a powerful argument for completing the 3-dimensional Hadamard-regularization results of [21, 22] and fixing the value of the unknown parameter. We leave to future work the task of an exact calculation of the d-dimensional equations of motion, instead of the calculation of the first few terms in a Laurent expansion in ε around $d=3$, as done here. The first step towards such a calculation is taken in Appendix C, where we give the explicit expression of the basic quadratically non-linear Green function $g(\mathbf{x},\mathbf{y}_1,\mathbf{y}_2)$ in d dimensions.

The detailed way of computing the difference between dim. reg. and Hadamard's reg. will turn out to be significantly more intricate than in the ADM case. This added complexity has several sources. A first source of complexity is that the harmonic-gauge d-dimensional calculation will be seen to contain (as anticipated long ago [8]) poles, proportional to $(d-3)^{-1}$,

[5] Thus the "kinetic ambiguity" parameter ω_k, originally introduced in the ADM approach [16, 17], takes the unique value $\omega_k = \frac{41}{24}$. This value was obtained in [21] using the result for the binary energy function in the case of circular orbits, as calculated in the harmonic-coordinates formalism, and also directly from the requirement of Poincaré invariance in the ADM formalism [19].

by contrast to the ADM calculation which is finite as $d \to 3$. A second source of complexity is that the end results [22] for the 3-dimensional 3PN equations of motion have been derived using systematically an *extended* version of the Hadamard regularization method, incorporating both a generalized theory of singular pseudo-functions and their associated (generalized) distributional derivatives [23], and an improved definition of the finite part as $\mathbf{x} \to \mathbf{y}_1$, say $[F]_1$, of a singular function $F(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$, designed so as to respect the global Poincaré symmetry of the problem [24]. We shall then find it technically convenient to subtract the various contributions to the end results of [22] which arose because of the specific use of the extended Hadamard regularization methods of [23, 24] before considering the difference with the d-dimensional result. A third source of added complexity (with respect to the ADM case [6]) comes from the presence in the harmonic-gauge integrals we shall evaluate of "hidden-distributional" terms in the integrands. By hidden distributional terms we mean terms proportional to the second spatial derivatives of the Poisson kernel $\Delta^{-1}\,\delta_a^{(d)} \propto r_a^{2-d}$, or to the fourth spatial derivatives of the iterated Poisson kernel $\Delta^{-2}\,\delta_a^{(d)} \propto r_a^{4-d}$. Such terms, $\partial_{ij}\,r_a^{2-d}$ or $\partial_{ijkl}\,r_a^{4-d}$, considered as Schwartz distributional derivatives [36], contain pieces proportional to the delta function $\delta_a^{(d)}$, which need to be treated with care. The generalized distributional derivative defined in [23], and used to compute the end results of [22], led to an improved way, compared to the normal Schwartz distributional derivative, of evaluating contributions coming from the product of a singular function and a derivative of the type $\partial_{ij}\,r_a^{-1}$ or $\partial_{ijkl}\,r_a$, and more generally of any derivatives of singular functions in a certain class. We shall find it convenient to subtract these additional non-Schwartzian contributions to the 3PN equations of motion before applying dimensional regularization. However, we shall note at the end that dim. reg. automatically incorporates all of these non-Schwartzian contributions.

A fourth, but minor, source of complexity concerns the dependence of the end results of [22] for the 3PN acceleration of the first particle (label $a = 1$), say $\mathbf{a}_1^{\mathrm{BF}}$, on two arbitrary length scales r'_1 and r'_2, and on the "ambiguity" parameter λ. Explicitly, we define

$$\mathbf{a}_1^{\mathrm{BF}}\left[\lambda; r'_1, r'_2\right] \equiv \text{R.H.S. of Eq. (7.16) in Ref. [22]}\,. \tag{1.5}$$

Here the acceleration is considered as a function of the two masses m_1 and m_2, the relative distance $\mathbf{y}_1 - \mathbf{y}_2 \equiv r_{12}\mathbf{n}_{12}$ (where $\mathbf{n}_{12}$ is the unit vector directed from particle 2 to particle 1), the two coordinate velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, and also, as emphasized in (1.5), the parameter λ as well as two regularization length scales r'_1 and r'_2. The latter length scales enter the equations of motion at the 3PN level through the logarithms $\ln(r_{12}/r'_1)$ and $\ln(r_{12}/r'_2)$. They come from the regularization as the field point $\mathbf{x}'$ tends to $\mathbf{y}_1$ or $\mathbf{y}_2$ of Poisson-type integrals (see Section III B below). The length scales r'_1, r'_2 are "pure gauge" in the sense that they can be removed by the effect induced on the world-lines of a coordinate transformation of the bulk metric [22]. On the other hand, the dimensionless parameter λ entering the final

[6] The specific form of the 3PN ADM Hamiltonian H derived in [18] and used (in its d-dimensional generalization) in [35] was written, on purpose, in a way which does not involve any hidden distributional terms (the only delta-function contributions it contains being explicit contact terms $F(\mathbf{x})\,\delta_a^{(3)}$). This allowed one to estimate the difference between the d-dimensional Hamiltonian $H^{(d)}$ and the Hadamard-regularized 3-dimensional one $\mathrm{Hr}[H^{(3)}]$ without having to worry about distributional derivatives. However, as a check on the consistency of dim. reg., the authors of [35] did perform another calculation of H based on a starting form of the Hamiltonian which involved hidden distributional terms, with the same final result.

result (1.5) corresponds to genuine physical effects. It was introduced by requiring that the 3PN equations of motion admit a conserved energy (and more generally be derivable from a Lagrangian). This extra requirement imposed *two relations* between the two length scales r'_1, r'_2 and two other length scales s_1, s_2 entering originally into the formalism, namely the constants s_1 and s_2 parametrizing the Hadamard partie finie of an integral as defined by Eq. (3.4) below. These relations were found to be of the form

$$\ln\left(\frac{r'_2}{s_2}\right) = \frac{159}{308} + \lambda\frac{m_1+m_2}{m_2} \quad \text{and } 1 \leftrightarrow 2\,, \tag{1.6}$$

where the so introduced *single* dimensionless parameter λ has been proved to be a purely numerical coefficient (independent of the two masses). When estimating the difference between dim. reg. and Hadamard reg. it will be convenient to insert Eq. (1.6) into (1.5) and to reexpress the acceleration of particle 1 in terms of the *original* regularization length scales entering the Hadamard regularization of $\mathbf{a}_1$, which were in fact r'_1 and s_2. Thus we can consider alternatively

$$\mathbf{a}_1^{\mathrm{BF}}[r'_1, s_2] \equiv \mathbf{a}_1^{\mathrm{BF}}\left[\lambda; r'_1, r'_2(s_2,\lambda)\right] \quad \text{and } 1 \leftrightarrow 2\,, \tag{1.7}$$

where the regularization constants are subject to the constraints (1.6) [we will then check that the λ-dependence on the R.H.S. of (1.7) disappears when using Eq. (1.6) to replace r'_2 as a function of s_2 and λ].

Our strategy will consist of *two steps*. The *first step* consists of subtracting all the extra contributions to Eq. (1.5), or equivalently Eq. (1.7), which were specific consequences of the extended Hadamard regularization defined in [23, 24]. As we shall detail below, there are *seven* such extra contributions $\delta^A\mathbf{a}_1$, $A=1,\cdots,7$. Essentially, subtracting these contributions boils down to estimating the value of $\mathbf{a}_1$ that would be obtained by using a "pure" Hadamard regularization, together with Schwartz distributional derivatives. Such a "pure Hadamard-Schwartz" (pHS) acceleration was in fact essentially the result of the first stage of the calculation of $\mathbf{a}_1$, as reported in the (unpublished) thesis [37]. It is given by

$$\mathbf{a}_1^{\mathrm{pHS}}\left[r'_1, s_2\right] = \mathbf{a}_1^{\mathrm{BF}}[r'_1, s_2] - \sum_{A=1}^{7}\delta^A\mathbf{a}_1\,. \tag{1.8}$$

The *second step* of our method consists of evaluating the Laurent expansion, in powers of $\varepsilon = d-3$, of the *difference* between the d-dimensional and the pure Hadamard-Schwartz (3-dimensional) computations of the acceleration $\mathbf{a}_1$. We shall see that this difference makes a contribution only when a term generates a *pole* $\sim 1/\varepsilon$, in which case dim. reg. adds an extra contribution, made of the pole and the finite part associated with the pole [we consistently neglect all terms $\mathcal{O}(\varepsilon)$]. One must then be especially wary of combinations of terms whose pole parts finally cancel ("cancelled poles") but whose dimensionally regularized finite parts generally do not, and must be evaluated with care. We denote the above defined difference

$$\mathcal{D}\mathbf{a}_1 = \mathcal{D}\mathbf{a}_1\left[\varepsilon, \ell_0; r'_1, s_2\right] \equiv \mathcal{D}\mathbf{a}_1\left[\varepsilon, \ell_0; \lambda; r'_1, r'_2\right]. \tag{1.9}$$

It depends both on the Hadamard regularization scales r'_1 and s_2 (or equivalently on λ and r'_1, r'_2) and on the regularizing parameters of dimensional regularization, namely ε and the characteristic length ℓ_0 associated with dim. reg. and introduced in Eq. (2.4) below. We shall explain in detail below the techniques we have used to compute $\mathcal{D}\mathbf{a}_1$ (see Section IV).

Finally, our main result will be the explicit computation of the ε-expansion of the dim. reg. acceleration as

$$\mathbf{a}_1^{\rm dr}[\varepsilon, \ell_0] = \mathbf{a}_1^{\rm pHS}[r'_1, s_2] + \mathcal{D}\mathbf{a}_1[\varepsilon, \ell_0; r'_1, s_2]\,. \tag{1.10}$$

With this result in hands, we shall prove (in Section VI) two theorems.

Theorem 1 *The pole part $\propto 1/\varepsilon$ of the dimensionally-regularized acceleration (1.10), as well as of the metric field $g_{\mu\nu}(x)$ outside the particles, can be re-absorbed (i.e., renormalized away) into some shifts of the two "bare" world-lines: $\mathbf{y}_a \to \mathbf{y}_a + \boldsymbol{\xi}_a$, with, say, $\boldsymbol{\xi}_a \propto 1/\varepsilon$ ("minimal subtraction"; MS), so that the result, expressed in terms of the "dressed" quantities, is finite when $\varepsilon \to 0$.*

The situation in harmonic coordinates is to be contrasted with the calculation in ADM-type coordinates within the Hamiltonian formalism [35], where it was shown that all pole parts directly cancel out in the total 3PN Hamiltonian (no shifts of the world-lines were needed). The central result of the paper is then as follows.

Theorem 2 *The "renormalized" (finite) dimensionally-regularized acceleration is physically equivalent to the extended-Hadamard-regularized acceleration (end result of Ref. [22]), in the sense that there exist some shift vectors $\boldsymbol{\xi}_1(\varepsilon, \ell_0; r'_1)$ and $\boldsymbol{\xi}_2(\varepsilon, \ell_0; r'_2)$, such that*

$$\mathbf{a}_1^{\rm BF}[\lambda, r'_1, r'_2] = \lim_{\varepsilon\to 0}\left[\mathbf{a}_1^{\rm dr}[\varepsilon, \ell_0] + \delta_{\boldsymbol{\xi}(\varepsilon,\ell_0;r'_1,r'_2)}\,\mathbf{a}_1\right] \tag{1.11}$$

(where $\delta_{\boldsymbol{\xi}}\,\mathbf{a}_1$ denotes the effect of the shifts on the acceleration[7]), if and only if the heretofore unknown parameter λ entering the harmonic-coordinates equations of motion takes the value

$$\lambda^{\text{dim. reg. harmonic}} = -\frac{1987}{3080}\,. \tag{1.12}$$

The precise shifts $\boldsymbol{\xi}_a(\varepsilon)$ needed in Theorem 2 involve not only a pole contribution $\propto 1/\varepsilon$, which defines the "minimal" (MS) shifts considered in Theorem 1, but also a finite contribution when $\varepsilon \to 0$. Their explicit expressions read:

$$\boldsymbol{\xi}_1 = \frac{11}{3}\frac{G_N^2\,m_1^2}{c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{r'_1\overline{q}^{1/2}}{\ell_0}\right) - \frac{327}{1540}\right]\mathbf{a}_{N1} \quad \text{and} \quad 1 \leftrightarrow 2\,, \tag{1.13}$$

where G_N is the usual Newton's constant [see Eq. (2.4) below], $\mathbf{a}_{N1}$ denotes the acceleration of the particle 1 (in d dimensions) at the Newtonian level, and $\overline{q} \equiv 4\pi e^C$ depends on the Euler constant $C = 0.577\cdots$. [The detailed proofs of Theorems 1 and 2 will consist of our investigations expounded in the successive sections of the paper, and will be completed at the end of Sections VI B and VI C respectively, taking also into account the results of Section VI D.]

Notice that an alternative way of presenting our central result is to say that, in fact, each choice of specific renormalization prescription (within dim. reg.), such as "minimal

[7] When working at the level of the equations of motion (not considering the metric outside the world-lines), the effect of shifts can be seen as being induced by a coordinate transformation of the bulk metric as in Ref. [22] (we comment on this point in Section VI B below).

subtraction" as assumed in Theorem 1 for conceptual simplicity,[8] leads to renormalized equations of motion which depend only on the dim. reg. characteristic length scale ℓ_0 through the logarithm $\ln(r_{12}/\ell_0)$, and that any of these renormalized equations of motion are physically equivalent to the final results of [22]. In particular, this means, as we shall see below, that each choice of renormalization prescription within dim. reg. determines the two regularization length scales r'_1, r'_2 entering Eq. (1.5). Of course, what is important is not the particular values these constants can take in a particular renormalization scheme [indeed r'_1 and r'_2 are simply "gauge" constants which can anyway be removed by a coordinate transformation], but the fact that the different renormalization prescriptions yield equations of motion falling into the "parametric" class (*i.e.*, parametrized by r'_1 and r'_2) of equations of motion obtained in [22].

An alternative way to phrase the result (1.11)-(1.12), is to combine Eqs. (1.8) and (1.10) in order to arrive at

$$\lim_{\varepsilon\to 0}\left[\mathcal{D}\mathbf{a}_1\left[\varepsilon,\ell_0;-\tfrac{1987}{3080};r'_1,r'_2\right]+\delta_{\boldsymbol{\xi}(\varepsilon,\ell_0;r'_1,r'_2)}\,\mathbf{a}_1\right]=\sum_{A=1}^{7}\delta^A\mathbf{a}_1\,. \tag{1.14}$$

Under this form one sees that the sum of the additional terms $\delta^A\mathbf{a}_1$ differs by a mere shift, *when and only when* λ takes the value (1.12), from the specific contribution $\mathcal{D}\mathbf{a}_1$ we shall evaluate in this paper, which comes directly from dimensional regularization. Therefore one can say that, when $\lambda=-\frac{1987}{3080}$, the extended-Hadamard regularization [23, 24] is in fact (physically) equivalent to dimensional regularization. However the extended-Hadamard regularization is incomplete, both because it is unable to determine λ, and also because it necessitates some "external" requirements such as the imposition of the link (1.6) in order to ensure the existence of a conserved energy — and in fact of the ten first integrals linked to the Poincaré group. By contrast dim. reg. succeeds automatically (without extra inputs) in guaranteeing the existence of the ten conserved integrals of the Poincaré group, as already found in Ref. [35].

In view of the necessary link (1.1) provided by the equivalence between the ADM-Hamiltonian and the harmonic-coordinates equations of motion, our result (1.12) is in perfect agreement with the previous result (1.4) obtained in [35].[9] Our result is also in agreement with the recent finding of Itoh and Futamase [38, 39] (see also [14]), who derived the 3PN equations of motion in harmonic gauge using a "surface-integral" approach, aimed at describing *extended* relativistic compact binary systems in the strong-field point particle limit. The surface-integral approach of Refs. [38, 39] is interesting because, like the matching method used at 2.5PN order in [8], it is based on the physical notion of extended compact bodies.

[8] However, for technical simplicity we shall prefer in Section VI C below to use a modified minimal subtraction that we shall denote $\boldsymbol{\xi}_{\overline{\rm MS}}$.

[9] One may wonder why the value of λ is a complicated rational fraction while ω_s is so simple. This is because ω_s was introduced precisely to measure the amount of ambiguities of certain integrals, and that the ADM Hamiltonian reported in [18] was put in a minimally ambiguous form, already in three dimensions, for which an *a posteriori* look at the "ambiguities" discussed in the Appendix A of [18] already showed that $\omega_s=0$. By contrast, λ has been introduced as the only possible unknown constant in the link between the four arbitrary scales r'_1, r'_2, s_1, s_2 (which has *a priori* nothing to do with ambiguities of integrals), in a framework where the use of the extended Hadamard regularization makes in fact the calculation to be unambiguous.

In this respect, we recall that the matching method used in [8] showed that the internal structure (Love numbers) of the constituent bodies would start influencing the equations of motion of (non-spinning) compact bodies only at the 5PN level. This *effacement property* strongly suggests that it is possible to model, in a physically preferred manner, two compact bodies as being two point-like particles, described by two masses and two world-lines, up to the 4.5PN level included. It remains, however, to prove that the dimensional regularization of delta-function sources does yield the physically unique equations of motion of two compact bodies up to the 4.5 PN order. The work [8] proved it at the 2.5 PN level, and the agreement of the present results with those of [38, 39] indicates that this is also true at the 3PN level.

Besides the independent confirmation of the value of ω_s or λ, let us also mention that our work provides a confirmation of the *consistency* of dim. reg., because our explicit calculations [which involved combinations of hundreds of Laurent expansions of the form $a_{-1}\,\varepsilon^{-1}+a_0+\mathcal{O}(\varepsilon)$] are entirely different from the ones of [35]: We use harmonic coordinates (instead of ADM-type ones), we work at the level of the equations of motion (instead of the Hamiltonian), we use a different form of Einstein's field equations and we solve them by a different iteration scheme.

Finally, from a practical point of view our confirmation of the value of ω_s orλ allows one to use the full 3PN accuracy in the analytical computation of the dynamics of the last orbits of binary orbits [40, 41]. It remains, however, the task of computing, using dimensional regularization, the parameter θ entering the 3.5PN gravitational energy flux [27, 28] to be able to have full 3.5PN accuracy in the computation of the gravitational waveforms emitted by inspiralling compact binaries (see, *e.g.*, [42] and references therein).

The organization of the paper is as follows. In Section II we derive our basic 3PN solution of the field equations for general fluid sources in d spatial dimensions, using d-dimensional generalizations of the elementary potentials introduced in Ref. [22]. Section III collects all the additional terms included in [22] which are due specifically to the extended Hadamard regularization, and derives the pure Hadamard-Schwartz (pHS) contribution to the equations of motion. The differences between the dimensional and pHS regularizations for all the potentials and their gradients are computed in Section IV. Then the dim. reg. equations of motion are obtained in Section V, where we comment also on their interpretation in terms of space-time diagrams. Section VI is devoted to the renormalization of the dim. reg. equations by means of suitable shifts of the particles' world-lines, and to the equivalence with the end results of [22] when Eq. (1.12) holds. At this stage, the proofs of Theorems 1 and 2 stated above are finally completed.

We end the paper with some conclusions (Section VII) and three appendices. Appendix A provides further material on the d-dimensional metric and geodesic equation, Appendix B gives a compendium of useful formulae for working in d dimensions, and Appendix C generalizes the well-known quadratic-order elementary kernel $\mathrm{g}^{(d=3)}(\mathbf{x}) = \ln(r_1+r_2+r_{12})$ to d dimensions. The latter calculation of the d-dimensional kernel $\mathrm{g}^{(d)}$ is not directly employed in the present paper, but represents a first step in obtaining the equations of motion in any dimension d (not necessarily of the form $3+\varepsilon$).

II. FIELD EQUATIONS IN $d+1$ SPACE-TIME DIMENSIONS

This section is devoted to the field equations of general relativity in $d+1$ space-time dimensions, and to the geodesic equation describing the motion of point particles. We use the sign conventions of Ref. [43], and in particular our metric signature is mostly $+$. Space-time indices are denoted by greek letters, and spatial indices by latin letters ($i, j, \cdots$ run from 1 to d). A summation is understood for any pair of repeated indices. We work in the harmonic gauge, which is such that

$$\Gamma^\lambda \equiv g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = 0\,. \tag{2.1}$$

As usual, $g^{\alpha\beta}$ denotes the inverse metric and $\Gamma^\lambda_{\alpha\beta}$ the Christoffel symbols. Using this gauge condition, one can easily prove that the Ricci tensor reads *in any dimension*

$$\begin{aligned}2R^{\rm harm}_{\mu\nu} \;=\; & -g^{\alpha\beta}g_{\mu\nu,\alpha\beta} + g^{\alpha\beta}g^{\gamma\delta}\biggl(g_{\mu\alpha,\gamma}\,g_{\nu\beta,\delta} - g_{\mu\alpha,\gamma}\,g_{\nu\delta,\beta} \\ & + g_{\mu\alpha,\gamma}\,g_{\beta\delta,\nu} + g_{\nu\alpha,\gamma}\,g_{\beta\delta,\mu} - \frac{1}{2}\,g_{\alpha\gamma,\mu}\,g_{\beta\delta,\nu}\biggr)\,,\end{aligned} \tag{2.2}$$

where a comma denotes partial derivation. Note that the spatial dimension d does not appear explicitly in this expression, whereas some d-dependent coefficients do appear when expressing the Ricci tensor in terms of the so-called "gothic" metric $\mathfrak{g}^{\mu\nu} \equiv \sqrt{-g}\,g^{\mu\nu}$ [see Eq. (A9) in Appendix A below].

In any dimension, the Einstein field equations read

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}\,T^{\mu\nu}\,, \tag{2.3}$$

where $T^{\mu\nu}$ denotes the matter stress-energy tensor, given by the functional derivative $\sqrt{-g}\,T^{\mu\nu} \equiv 2c\,\delta S_m/\delta g_{\mu\nu}$ of the matter action S_m with respect to the metric tensor. *By definition*, G denotes the constant involved in Eq. (2.3), which shows that its dimension is such that

$$G = G_N \ell_0^{d-3}\,, \tag{2.4}$$

where G_N is the usual Newton constant (in 3 spatial dimensions) and ℓ_0 is an arbitrary length scale. This scale will be involved in our dimensionally regularized results below, but we will finally show that the physical observables do not depend on it. As is well known, the combination of Eq. (2.3) with its trace allows us to rewrite it as

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{d-1}\,g_{\mu\nu}T^\lambda_\lambda\right)\,, \tag{2.5}$$

in which the spatial dimension d now appears explicitly.

We wish to expand in powers of $1/c$ the field equations resulting from (2.2) and (2.5). The basic idea is to introduce a sequence of "elementary potentials", V, V_i, $\hat{W}_{ij}$, ... which allow one to parametrize conveniently the successive post-Minkowskian contributions to the metric $g_{\mu\nu}(x)$. For instance, at the first post-Minkowskian order it is convenient to parametrize the metric as

$$g_{00} \equiv -1 + 2V/c^2 + \mathcal{O}(G^2)\,, \quad g_{0i} \equiv -4V_i/c^3 + \mathcal{O}(G^2)\,, \tag{2.6}$$

where the so-introduced elementary potentials V and V_i satisfy equations of the form

$$\Box V = -4\pi G\sigma\,, \quad \Box V_i = -4\pi G\sigma_i\ , \tag{2.7}$$

where $\Box \equiv \partial_i^2 - (1/c^2)\partial_t^2$ denotes the flat d'Alembertian and where, *by definition*, the sources σ and σ_i are linear combinations of the contravariant components $T^{\mu\nu}$ of the stress-energy tensor of the matter. Let us underline that the factor $-4\pi G$ in these equations is a *choice*. We could of course introduce here a functional dependence on the spatial dimension d, for instance by replacing the factor 4π by the surface of the unit $(d-1)$-dimensional sphere [see Eq. (B3) in Appendix B], but this would only complicate the intermediate expressions without changing our final result. The matter sources σ and σ_i *defined* by Eqs. (2.6), (2.7) read (in d spatial dimensions):

$$\sigma \equiv \frac{2}{d-1}\,\frac{(d-2)T^{00}+T^{ii}}{c^2}\,, \qquad \sigma_i \equiv \frac{T^{0i}}{c}\ , \qquad \sigma_{ij} \equiv T^{ij}\,. \tag{2.8}$$

The definition for σ_{ij} has been added for future use. Note that σ_i and σ_{ij} take the same forms as usual in 3 dimensions (see Eqs. (3.9) of Ref. [22]), but that the definition of σ involves an explicit dependence on d. Conversely, the first and third of these equations allow us to express T^{00} in terms of the above matter sources: $T^{00} = \left(\frac{d-1}{2}\,\sigma c^2 - \sigma_{ii}\right)/(d-2)$. A simple consequence of the expression of σ is that the d-dimensional *Newtonian potential* generated by a mass m_a located at $\mathbf{y}_a$ reads explicitly

$$U_a(\mathbf{x}) = V_a(\mathbf{x}) + \mathcal{O}\left(\frac{1}{c^2}\right) = 2\,\frac{d-2}{d-1}\,\tilde{k}\,\frac{G\,m_a}{r_a^{d-2}} + \mathcal{O}\left(\frac{1}{c^2}\right)\,, \tag{2.9}$$

where the factor $2(d-2)/(d-1)$ comes from σ (*i.e.*, from Einstein's equations), while the factor $\tilde{k} = \Gamma\left(\frac{d-2}{2}\right)/\pi^{\frac{d-2}{2}}$ comes from the expression of the Green function of the Laplacian in d dimensions (see Eq. (4.12) below and Appendix B).

We give below the simplest forms of the metric and of the potential equations that we could obtain. We will explain afterwards which rules we followed to simplify them. Let us first define the useful combination

$$\mathcal{V} \equiv V - \frac{2}{c^2}\left(\frac{d-3}{d-2}\right)K + \frac{4\hat{X}}{c^4} + \frac{16\hat{T}}{c^6}\,. \tag{2.10}$$

Then the metric components can be written in a rather compact form:

$$g_{00} = -e^{-2\mathcal{V}/c^2}\left(1 - \frac{8V_iV_i}{c^6} - \frac{32\hat{R}_iV_i}{c^8}\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right), \tag{2.11a}$$

$$g_{0i} = -e^{-\frac{(d-3)\mathcal{V}}{(d-2)c^2}}\left\{\frac{4V_i}{c^3}\left[1 + \frac{1}{2}\left(\frac{d-1}{d-2}\,\frac{V}{c^2}\right)^2\right] + \frac{8\hat{R}_i}{c^5} + \frac{16}{c^7}\left[\hat{Y}_i + \frac{1}{2}\hat{W}_{ij}V_j\right]\right\} + \mathcal{O}\left(\frac{1}{c^9}\right), \tag{2.11b}$$

$$g_{ij} = e^{\frac{2\mathcal{V}}{(d-2)c^2}}\left\{\delta_{ij} + \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left[\hat{Z}_{ij} - V_iV_j + \frac{1}{2(d-2)}\,\delta_{ij}V_kV_k\right]\right\} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{2.11c}$$

The various elementary potentials V, V_i, K, $\hat{W}_{ij}$, $\hat{R}_i$, $\hat{X}$, $\hat{Z}_{ij}$, $\hat{Y}_i$ and $\hat{T}$ introduced in these definitions are d-dimensional analogues of those used in Eqs. (3.24) of Ref. [22]. Actually,

an extra potential is needed for $d \neq 3$, and it has been denoted K in Eq. (2.10) above. We give in Appendix A the explicit expansion of this metric in powers of $1/c$, as well as its inverse $g^{\mu\nu}$ and its determinant g, which can be useful for future works. Note that the first post-Newtonian order of the spatial metric, $g_{ij} = \delta_{ij}\big[1 + \frac{2V}{(d-2)c^2}\big] + \mathcal{O}(1/c^4)$, explicitly depends on d contrary to our choice (2.6) for g_{00}. This dissymmetry between g_{00} and g_{ij} is imposed by the field equations (2.5).

The successive post-Newtonian truncations of the field equations (2.2)-(2.5) give us the sources for these various potentials. The equations for $\Box V$ and $\Box V_i$ have already been written in Eqs. (2.7) above. We get for the remaining potentials:

$$\Box K = -4\pi G \sigma V \,, \tag{2.12a}$$

$$\Box \hat{W}_{ij} = -4\pi G\left(\sigma_{ij} - \delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right) - \frac{1}{2}\left(\frac{d-1}{d-2}\right)\partial_i V \partial_j V \,, \tag{2.12b}$$

$$\begin{aligned}\Box \hat{R}_i = &-\frac{4\pi G}{d-2}\left(\frac{5-d}{2}\,V\sigma_i - \frac{d-1}{2}\,V_i\,\sigma\right)\\ &-\frac{d-1}{d-2}\,\partial_k V \partial_i V_k - \frac{d(d-1)}{4(d-2)^2}\,\partial_t V \partial_i V \,,\end{aligned} \tag{2.12c}$$

$$\begin{aligned}\Box \hat{X} = &-4\pi G\left[\frac{V\sigma_{ii}}{d-2} + 2\left(\frac{d-3}{d-1}\right)\sigma_i V_i + \left(\frac{d-3}{d-2}\right)^2 \sigma\left(\frac{V^2}{2} + K\right)\right]\\ &+\hat{W}_{ij}\,\partial_{ij}V + 2V_i\,\partial_t\partial_i V + \frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2 V\\ &+\frac{d(d-1)}{4(d-2)^2}\left(\partial_t V\right)^2 - 2\partial_i V_j\,\partial_j V_i \,,\end{aligned} \tag{2.12d}$$

$$\begin{aligned}\Box \hat{Z}_{ij} = &-\frac{4\pi G}{d-2}\,V\left(\sigma_{ij} - \delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right) - \frac{d-1}{d-2}\,\partial_t V_{(i}\,\partial_{j)}V + \partial_i V_k\,\partial_j V_k\\ &+\partial_k V_i\,\partial_k V_j - 2\partial_k V_{(i}\,\partial_{j)}V_k - \frac{\delta_{ij}}{d-2}\,\partial_k V_m\left(\partial_k V_m - \partial_m V_k\right)\\ &-\frac{d(d-1)}{8(d-2)^3}\,\delta_{ij}\left(\partial_t V\right)^2 + \frac{(d-1)(d-3)}{2(d-2)^2}\,\partial_{(i}V\partial_{j)}K \,,\end{aligned} \tag{2.12e}$$

$$\begin{aligned}\Box \hat{Y}_i = &-4\pi G\Bigg[-\frac{1}{2}\left(\frac{d-1}{d-2}\right)\sigma\hat{R}_i - \frac{(5-d)(d-1)}{4(d-2)^2}\,\sigma V V_i + \frac{1}{2}\,\sigma_k\hat{W}_{ik} + \frac{1}{2}\sigma_{ik}V_k\\ &\qquad + \frac{1}{2(d-2)}\,\sigma_{kk}V_i - \frac{d-3}{(d-2)^2}\,\sigma_i\left(V^2 + \frac{5-d}{2}\,K\right)\Bigg]\\ &+\hat{W}_{kl}\,\partial_{kl}V_i - \frac{1}{2}\left(\frac{d-1}{d-2}\right)\partial_t\hat{W}_{ik}\,\partial_k V + \partial_i\hat{W}_{kl}\,\partial_k V_l - \partial_k\hat{W}_{il}\,\partial_l V_k\\ &-\frac{d-1}{d-2}\,\partial_k V\partial_i\hat{R}_k - \frac{d(d-1)}{4(d-2)^2}\,V_k\,\partial_i V\partial_k V - \frac{d(d-1)^2}{8(d-2)^3}\,V\partial_t V\partial_i V\\ &-\frac{1}{2}\left(\frac{d-1}{d-2}\right)^2 V\partial_k V\partial_k V_i + \frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2 V_i + 2V_k\,\partial_k\partial_t V_i\\ &+\frac{(d-1)(d-3)}{(d-2)^2}\,\partial_k K\partial_i V_k + \frac{d(d-1)(d-3)}{4(d-2)^3}\left(\partial_t V\partial_i K + \partial_i V\partial_t K\right),\end{aligned} \tag{2.12f}$$

$$
\begin{aligned}
\Box\hat{T} \;=\; -4\pi G\bigg[& \frac{1}{2(d-1)}\,\sigma_{ij}\hat{W}_{ij} + \frac{5-d}{4(d-2)^2}\,V^2\sigma_{ii} + \frac{1}{d-2}\,\sigma V_i V_i - \frac{1}{2}\left(\frac{d-3}{d-2}\right)\sigma\hat{X} \\
& - \frac{1}{12}\left(\frac{d-3}{d-2}\right)^3 \sigma V^3 - \frac{1}{2}\left(\frac{d-3}{d-2}\right)^3 \sigma V K + \frac{(5-d)(d-3)}{2(d-1)(d-2)}\,\sigma_i V_i V \\
& + \frac{d-3}{d-1}\,\sigma_i\hat{R}_i - \frac{d-3}{2(d-2)^2}\,\sigma_{ii}K\bigg] + \hat{Z}_{ij}\,\partial_{ij}V + \hat{R}_i\,\partial_t\partial_i V \\
& -2\partial_i V_j\,\partial_j\hat{R}_i - \partial_i V_j\,\partial_t\hat{W}_{ij} + \frac{1}{2}\left(\frac{d-1}{d-2}\right) V V_i\,\partial_t\partial_i V + \frac{d-1}{d-2}\,V_i\,\partial_j V_i\,\partial_j V \\
& + \frac{d(d-1)}{4(d-2)^2}\,V_i\,\partial_t V\,\partial_i V + \frac{1}{8}\left(\frac{d-1}{d-2}\right)^2 V^2\partial_t^2 V + \frac{d(d-1)^2}{8(d-2)^3}\,V\,(\partial_t V)^2 \\
& - \frac{1}{2}\,(\partial_t V_i)^2 - \frac{(d-1)(d-3)}{4(d-2)^2}\,V\partial_t^2 K - \frac{d(d-1)(d-3)}{4(d-2)^3}\,\partial_t V\,\partial_t K \\
& - \frac{(d-1)(d-3)}{4(d-2)^2}\,K\partial_t^2 V - \frac{d-3}{d-2}\,V_i\,\partial_t\partial_i K - \frac{1}{2}\left(\frac{d-3}{d-2}\right)\hat{W}_{ij}\,\partial_{ij}K\,. \qquad (2.12\text{g})
\end{aligned}
$$

In Eq. (2.12e), parentheses around indices mean their symmetrization, *i.e.*, $a_{(ij)} \equiv \frac{1}{2}(a_{ij} + a_{ji})$. For $d = 3$, the above set of equations (2.12) reduces to Eqs. (3.26) and (3.27) of Ref. [22]. The order of the terms and their writing has been chosen to be as close as possible to this reference.

The harmonic gauge conditions (2.1) impose the following differential identities between the potentials:

$$
\begin{aligned}
g^{\mu\nu}\Gamma^0_{\mu\nu} = 0 \;\Rightarrow\; & \partial_t\bigg\{\frac{1}{2}\left(\frac{d-1}{d-2}\right)V + \frac{1}{2c^2}\left[\hat{W} + \left(\frac{d-1}{d-2}\right)^2 V^2 - \frac{2(d-1)(d-3)}{(d-2)^2}\,K\right] \\
& \quad + \frac{2}{c^4}\left(\frac{d-1}{d-2}\right)\bigg[\hat{X} + \frac{d-2}{d-1}\,\hat{Z} - \frac{d-3}{d-1}\,V_k V_k + \frac{1}{2}\,V\hat{W} \\
& \qquad + \frac{(d-1)^2}{6(d-2)}\,V^3 - \frac{(d-1)(d-3)}{(d-2)^2}\,VK\bigg]\bigg\} \\
& + \partial_i\bigg\{V_i + \frac{2}{c^2}\left[\hat{R}_i + \frac{1}{2}\left(\frac{d-1}{d-2}\right)V V_i\right] \\
& \quad + \frac{4}{c^4}\bigg[\hat{Y}_i - \frac{1}{2}\,\hat{W}_{ij}V_j + \frac{1}{2}\,\hat{W}V_i + \frac{1}{2}\left(\frac{d-1}{d-2}\right)V\hat{R}_i + \frac{1}{4}\left(\frac{d-1}{d-2}\right)^2 V^2 V_i \\
& \qquad - \frac{(d-1)(d-3)}{2(d-2)^2}\,K V_i\bigg]\bigg\} = \mathcal{O}\left(\frac{1}{c^6}\right), \qquad (2.13\text{a})
\end{aligned}
$$

$$
\begin{aligned}
g^{\mu\nu}\Gamma^i_{\mu\nu} = 0 \;\Rightarrow\; & \partial_t\left\{V_i + \frac{2}{c^2}\left[\hat{R}_i + \frac{1}{2}\left(\frac{d-1}{d-2}\right)V V_i\right]\right\} \\
& + \partial_j\left\{\hat{W}_{ij} - \frac{1}{2}\,\hat{W}\delta_{ij} + \frac{4}{c^2}\left[\hat{Z}_{ij} - \frac{1}{2}\,\hat{Z}\delta_{ij}\right]\right\} = \mathcal{O}\left(\frac{1}{c^4}\right), \qquad (2.13\text{b})
\end{aligned}
$$

where $\hat{W} \equiv \hat{W}_{kk}$ and $\hat{Z} \equiv \hat{Z}_{kk}$ denote the traces of potentials $\hat{W}_{ij}$ and $\hat{Z}_{ij}$. For $d = 3$, these identities reduce to Eqs. (3.28) of Ref. [22]. In this paper we shall check (see Sections IV A

and VI A) that all the dimensionally-regularized potentials we use obey, at the indicated accuracy, the differential identities (2.13) equivalent to the harmonic gauge conditions.

In order to simplify as much as possible the above equations (2.12) for the potentials, we used the following rules:

(i) We used the harmonic gauge condition (2.13a) to replace everywhere $\partial_i V_i$ in terms of $\partial_t V$ and higher post-Newtonian order terms, and the gauge condition (2.13b) to replace $\partial_j \hat{W}_{ij}$ in terms of $\partial_i \hat{W}$ and $\partial_t V_i$ [our knowledge of the higher order terms $\mathcal{O}(1/c^2)$ in Eq. (2.13b) was actually not necessary for the simplification of Eqs. (2.12)]. We also used the lowest order terms of Eqs. (2.13) to simplify their own higher order contributions.

(ii) If the source of a potential P contained a double (contracted) gradient of the form $\Box P = \partial_k A \partial_k B +$ (other terms), where A and B were two lower-order potentials, we got rid of the double gradient by defining another potential $P' \equiv P - \frac{1}{2}AB$. We could then write its equation as $\Box P' = -\frac{1}{2}(\Box A)B - \frac{1}{2}A(\Box B) + \frac{1}{c^2}\,\partial_t A \partial_t B +$ (other terms), in which $\Box A$ and $\Box B$ were replaced by their own explicit sources. The contribution proportional to $1/c^2$ was then transferred into the source of a higher order potential.

(iii) At order $\mathcal{O}(1/c^6)$, equation (2.2)-(2.5) for R_{00} (*i.e.*, for $\Box g_{00}$) contains the term $\hat{W}_{ij}\,\partial_{ij}V$, that we introduced in the source of potential $\hat{X}$, Eq. (2.12d). In all other equations involving the same source $\hat{W}_{ij}\,\partial_{ij}V$, we used $\Box\hat{X}$ to eliminate it, instead of reintroducing it in the sources of other potentials. This is the reason why $\hat{X}$ is involved in the spatial metric g_{ij} too at order $\mathcal{O}(1/c^6)$ [*via* the exponential of $\mathcal{V}$ in Eq. (2.11c)], and why $V\hat{X}$ appears again in g_{00} at order $\mathcal{O}(1/c^8)$. See the expanded form of the metric (A1) in Appendix A.

(iv) In the equation for R_{00} at order $\mathcal{O}(1/c^8)$, we *chose* to eliminate a source proportional to $V\partial_i V_j\,\partial_i V_j$, by including an all-integrated term VV_iV_i in the definition of g_{00}, Eq. (2.11a). On the other hand, we could not eliminate at the same time the source term proportional to $V_i\,\partial_j V_i\,\partial_j V$ in $\Box g_{00}$, although it involves a double (contracted) gradient too. This is the reason why such a term appears in Eq. (2.12g).

The above simplification rules have been applied systematically with a single exception. Indeed, Eq. (2.12e) for $\Box\hat{Z}_{ij}$ involves double (contracted) gradients $\partial_k V_i\,\partial_k V_j$ and $\delta_{ij}\,\partial_k V_m\,\partial_k V_m$. Therefore, the application of rule (ii) would have yielded another potential

$$\widetilde{Z}_{ij} \equiv \hat{Z}_{ij} - \frac{1}{2}V_iV_j + \frac{1}{2(d-2)}\,\delta_{ij}V_kV_k\ , \tag{2.14}$$

such that no double gradient appears in its source (but extra compact sources $\sigma_{(i}V_{j)}$ and $\delta_{ij}\sigma_k V_k$ would have been involved). Although this modified potential $\widetilde{Z}_{ij}$ actually simplifies slightly some equations (but not all of them), we have chosen to use $\hat{Z}_{ij}$ which is the direct d-dimensional analogue of the potential written in Eq. (3.27c) of Ref. [22]. Indeed, as explained in the following sections, the 3-dimensional results of this reference will be necessary for our d-dimensional calculations, and it is more convenient to keep the same notation.

Notice also that after the above simplifications, the resulting metric involves only potentials which are at most cubically non-linear (like for the term $\hat{W}_{ij}\,\partial_{ij}V$ in the potential $\hat{X}$ — using the terminology of Section V C below). There is no need to introduce any quartically

non-linear elementary potential because it turns out that it is possible to "integrate directly" all of them (at the 3PN level) in terms of other potentials. The only quartic contributions are the terms composed of V^4 and $V\hat{X}$ in the metric component g_{00} [see Eq. (A1a) in Appendix A]. The fact that there are no intrinsically quartic potentials at the 3PN order made the closed-form calculation in [21, 22] possible. We shall comment more on this interesting fact in Section IV A.

Let us now apply the general potential parametrization of the metric defined above to the specific case of (monopolar) point particles, *i.e.*, to the action

$$S = \int \frac{d^{d+1}x}{c}\sqrt{-g}\,\frac{c^4}{16\pi G}\,R(g) - \sum_a m_a\,c\int\sqrt{-g_{\mu\nu}(y_a^\lambda)\,dy_a^\mu\,dy_a^\nu}\,. \tag{2.15}$$

The stress-energy tensor $T^{\mu\nu}(x) \equiv [2c/\sqrt{-g(x)}]\,\delta\,S_{\rm matter}/\delta\,g_{\mu\nu}(x)$ deduced from this action reads

$$T^{\mu\nu}(x) = \sum_a m_a\,c^2\int ds_a\,\frac{dy_a^\mu}{ds_a}\,\frac{dy_a^\nu}{ds_a}\,[-g(y_a)]^{-\frac{1}{2}}\,\delta^{(d+1)}(x^\lambda - y_a^\lambda(s_a))\,, \tag{2.16}$$

where $ds_a \equiv \sqrt{-g_{\mu\nu}\,(y_a^\lambda)\,dy_a^\mu\,dy_a^\nu}$ is (c times) the proper time along the world-line of the $a^{\rm th}$ particle and where $\delta^{(d+1)}$ is the Dirac density in $d+1$ dimensions ($\int d^{d+1}x\,\delta^{(d+1)}(x) = 1$). Here, we take advantage of the fact (emphasized in [35]) that dim. reg. respects the basic properties of the algebraic and differential calculus: associativity, commutativity and distributivity of point-wise addition and multiplication, Leibniz's rule, Schwarz's rule ($\partial_\mu\,\partial_\nu\,f = \partial_\nu\,\partial_\mu\,f$), integration by parts, etc. In addition, the post-Newtonian expansion of $g_{\mu\nu}(x)$ yields "d-dimensional functions" which are formally as smooth as wished (by taking the real part of d small enough) in the vicinity of the world-lines: see for instance Eq. (2.9). This allows us to work with self-gravitating point particles essentially as if they were *test* particles. For instance, we can use $F[g_{\mu\nu}(x)]\,\delta^{(d)}\,(\mathbf{x}-\mathbf{y}_a) = F[g_{\mu\nu}(y_a)]\,\delta^{(d)}\,(\mathbf{x}-\mathbf{y}_a)$. In particular, the y_a-evaluated determinant factor $[-g(y_a)]^{-\frac{1}{2}}$ in (2.16) came from the field-point dependent factor $[-g(x)]^{-\frac{1}{2}}$ in the definition of $T^{\mu\nu}(x)$. Similarly, the usual derivation of the equations of motion of a test particle formally generalizes to the case of self-gravitating point particles in d dimensions. One then finds that the equations of motion of point particles can equivalently be written as

$$\nabla_\nu\,T^{\mu\nu}(x) = 0\,, \tag{2.17}$$

or as the usual geodesic equations. The latter can be written either in covariant form $u_a^\nu\,\nabla_\nu\,u_a^\mu = 0$ ($u_a^\mu \equiv dy_a^\mu/ds_a$), *i.e.*, explicitly

$$\frac{d^2\,y_a^\lambda}{ds_a^2} + \Gamma^\lambda_{\mu\nu}\,[g(y_a), \partial g(y_a)]\,\frac{dy_a^\mu}{ds_a}\,\frac{dy_a^\nu}{ds_a} = 0\,, \tag{2.18}$$

where $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\,g^{\lambda\sigma}(\partial_\mu\,g_{\nu\sigma} + \partial_\nu\,g_{\mu\sigma} - \partial_\sigma\,g_{\mu\nu})$ as usual, or in the explicit form corresponding to using the coordinate time $t = y_a^0/c$ as parameter along the world-lines, which is easily derived from the covariant expression with a lower index, $u_a^\nu\,\nabla_\nu\,u^a_\mu = 0 \Leftrightarrow d(g_{\mu\rho}u_a^\rho)/ds_a = \frac{1}{2}\partial_\mu g_{\nu\rho}\,u_a^\nu u_a^\rho$. Like in 3 dimensions, cf. Eqs. (3.32)-(3.33) of Ref. [22], it can thus be put in the form

$$\frac{dP^i}{dt} = F^i\,, \tag{2.19}$$

where

$$P^i \equiv \frac{g_{i\mu} v^\mu}{\sqrt{-g_{\rho\sigma} v^\rho v^\sigma / c^2}} \ , \quad F^i \equiv \frac{1}{2} \frac{\partial_i g_{\mu\nu} v^\mu v^\nu}{\sqrt{-g_{\rho\sigma} v^\rho v^\sigma / c^2}} \ , \tag{2.20}$$

$v^\mu \equiv dx^\mu/dt = (c, \mathbf{v})$ denoting the coordinate velocity. Let us emphasize again that in d dimensions, all the non-linear functions of $g_{\mu\nu}(y_a)$ and $\partial_\lambda \, g_{\mu\nu}(y_a)$ that will enter our calculation of (2.19)-(2.20) can be treated as in the $\mathbf{x} \to \mathbf{y}_a$ evaluation of smooth functions of $\mathbf{x}$. For instance, denoting for simplicity $f \equiv 2(d-2)/(d-1)$, the Newtonian approximation, say $U^{(d)}(\mathbf{x}) \equiv U(\mathbf{x})$, of the basic scalar potential $V(\mathbf{x})$, reads, in the vicinity of $\mathbf{x} = \mathbf{y}_1$,

$$U(\mathbf{x}) = f \, \tilde{k} \, G \, m_1 \, r_1^{2-d} + U_2(\mathbf{x}) \, , \tag{2.21}$$

where $U_2(\mathbf{x}) = f \, \tilde{k} \, G \, m_2 \, r_2^{2-d}$ is (in any d) an indefinitely differentiable function of $\mathbf{x}$ near $\mathbf{y}_1$. Analytically continuing d to sufficiently "low" (and even with negative real part, if needed) values, we see not only that $[U(\mathbf{x})]_{\mathbf{x}=\mathbf{y}_1} = U_2(\mathbf{y}_1)$, but that $[U^n(\mathbf{x})]_{\mathbf{x}=\mathbf{y}_1} = (U_2(\mathbf{y}_1))^n$, and, *e.g.*, $[U^p(\mathbf{x}) \, \partial_i \, U(\mathbf{x})]_{\mathbf{x}=\mathbf{y}_1} = (U_2(\mathbf{y}_1))^p \, \partial_i \, U_2(\mathbf{y}_1)$, etc.

Although the expressions (2.20) do not depend explicitly on the dimension d, the metric (2.11) does depend on it, and therefore the post-Newtonian expansions of Eqs. (2.20) involve many d-dependent coefficients. We give their full expressions in Appendix A, Eqs. (A11)-(A12), but we quote below only their Newtonian orders and the very few terms which will contribute to the poles $\propto 1/(d-3)$ in our dimensionally regularized calculations:

$$P^i = v^i + \cdots - \frac{8}{c^4} \hat{R}_i - \frac{16}{c^6} \hat{Y}_i + \mathcal{O}\left(\frac{1}{c^8}\right) , \quad F^i = \partial_i V + \cdots + \frac{4}{c^4} \partial_i \hat{X} + \frac{16}{c^6} \partial_i \hat{T} + \mathcal{O}\left(\frac{1}{c^8}\right) . \tag{2.22}$$

The acceleration $\mathbf{a} \equiv d\mathbf{v}/dt$ can thus be written as

$$\begin{aligned} a^i \ &= \ F^i - \frac{d\,(P^i - v^i)}{dt} \qquad\qquad (2.23) \\ &= \ \partial_i V + \frac{1}{c^2} \Big[\cdots \Big] + \frac{4}{c^4} \left[\partial_i \hat{X} + 2 \frac{d \hat{R}_i}{dt} + \cdots \right] + \frac{16}{c^6} \left[\partial_i \hat{T} + \frac{d \hat{Y}_i}{dt} + \cdots \right] + \mathcal{O}\left(\frac{1}{c^8}\right) . \end{aligned}$$

In Section IV A we shall give flesh to the formal expressions written above by explaining by which algorithm one can compute, with the required accuracy, the explicit d-dimensional expansions near $\mathbf{x} = \mathbf{y}_1$ [analogous to the simple case (2.21)] of the various elementary potentials entering Eq. (2.23), and notably of the crucial ones $\hat{X}$, $\hat{R}_i$ (to be computed with 1PN accuracy) and $\hat{T}$, $\hat{Y}_i$ (to be computed at Newtonian order only).

III. HADAMARD SELF-FIELD REGULARIZATIONS IN 3 DIMENSIONS

The main aim of this Section is to complete the *first step* of the strategy outlined in the Introduction, *i.e.* to collect a complete list of the additional contributions to the equations of motion which are specific consequences of the use of the *extended* Hadamard regularization methods defined in [23, 24]. However, to do that we need to start by recalling some material concerning the Hadamard regularization in 3 dimensions, and by contrasting it with dimensional regularization. Such material is needed for understanding our computation based on the "difference" in Section IV. We shall start by recalling the definition of

the "ordinary" Hadamard regularization and complete it by defining what we shall call the "pure" Hadamard regularization. Then we shall recall the main new features of the *extended* Hadamard regularization defined in [23, 24], and collect the additional contributions to the equations of motion which are specific consequences of the use of the extended Hadamard regularization (there are seven such additional contributions).

A. Ordinary and "pure" Hadamard regularizations

The phrase "Hadamard regularization" covers two distinct concepts: (i) the regularization of the "limit" $\lim_{\mathbf{x}\to\mathbf{y}_1} F(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)$ where $F(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)$ belongs to a class $\mathcal{F}$ of singular functions (generated by the iteration of Einstein's equations), and (ii) the regularization of the 3-dimensional integral $\int d^3\mathbf{x}\, F(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)$ of some function $F \in \mathcal{F}$. The class of functions $\mathcal{F}$ consists of all functions $F(\mathbf{x})$ on $\mathbb{R}^3$ that are smooth except at $\mathbf{y}_1$ and $\mathbf{y}_2$, around which they admit Laurent-type expansions in powers of r_1 or r_2 (see Section II.A of [23] for the precise definition of $\mathcal{F}$). When $r_1 \equiv |\mathbf{x}-\mathbf{y}_1| \to 0$ (*i.e.*, around singularity 1) we have, $\forall N \in \mathbb{N}$,

$$F(\mathbf{x}) = \sum_{p_0 \leq p \leq N} r_1^p \underset{1}{f}{}_p(\mathbf{n}_1) + o(r_1^N)\,, \tag{3.1}$$

where the Landau o-symbol takes its usual meaning, and the ${}_1f_p(\mathbf{n}_1)$'s denote the coefficients of the various powers of r_1, which are functions of the positions and velocities of the particles, and of the unit direction $\mathbf{n}_1 \equiv (\mathbf{x}-\mathbf{y}_1)/r_1$ of approach to singularity 1. The powers of r_1 are relative integers, $p \in \mathbb{Z}$, bounded from below by some typically negative p_0 depending on the F in question.

The Hadamard "*partie finie*" of the singular function F at the location of the singular point 1 (first meaning of Hadamard regularization) is defined as the angular average of the zeroth-order coefficient in the expansion (3.1). It is denoted $(F)_1$, so that

$$(F)_1 \equiv \left\langle \underset{1}{f}{}_0 \right\rangle \equiv \int \frac{d\Omega(\mathbf{n}_1)}{4\pi} \underset{1}{f}{}_0(\mathbf{n}_1)\,, \tag{3.2}$$

where $d\Omega(\mathbf{n}_1)$ denotes the usual surface element on the 2-dimensional sphere centered on 1. We shall employ systematically a bracket notation $\langle\rangle$ for the angular average of a function of the angles (either $\mathbf{n}_1$ or $\mathbf{n}_2$).[10] A distinctive feature of the Hadamard partie finie (3.2) is its "non-distributivity" in the sense that

$$(FG)_1 \neq (F)_1(G)_1 \;\;\text{in general for } F,G \in \mathcal{F}. \tag{3.3}$$

The non-distributivity represents a crucial departure away from the simple algebraic properties of the analog of $(F)_1$ in dim. reg. which is merely $F^{(d)}(\mathbf{y}_1)$. It is an interesting fact that in a post-Newtonian expansion the non-distributivity starts playing a role only at the 3PN order (because the functions there become singular enough). Up to the 2PN order one can show that $(FG)_1 = (F)_1(G)_1$ for all the functions involved in the equations of motion in harmonic coordinates [13]. Several of the problems of the Hadamard self-field regularization

[10] Since this will always be clear from the context, we do not specify on the brackets if the angular integration should be performed around point 1 or 2. We do not indicate either if the integration sphere is two-dimensional or $(d-1)$-dimensional (as we shall see later there can be no confusion about this).

(in the "ordinary" sense) when applied at the 3PN level (*e.g.* the occurrence of the unknown constant λ) are related to the latter non-distributivity.

The second notion of Hadamard *partie finie* (denoted Pf in the following) is to give a meaning to the generally divergent integral $\int d^3\mathbf{x}\, F(\mathbf{x})$. In this work we shall have to consider only the ultra-violet (UV) divergencies of the integrals, *i.e.*, at the locations of the two local singularities $\mathbf{y}_1$ and $\mathbf{y}_2$. All functions involved at the 3PN order are such that there are no infra-red (IR) divergencies when $|\mathbf{x}| \to \infty$ (this is true not only in 3 dimensions but also for any dimension $d = 3 + \varepsilon$ in a neighborhood of $d = 3$). The Hadamard partie finie of the (UV) divergencies is then defined as

$$\begin{aligned} \mathrm{Pf}_{s_1,s_2} \int d^3\mathbf{x}\, F(\mathbf{x}) \;\equiv\; & \lim_{s\to 0} \biggl\{ \int_{\mathbb{R}^3 \setminus B_1(s) \cup B_2(s)} d^3\mathbf{x}\, F(\mathbf{x}) \\ & + 4\pi \sum_{p+3<0} \frac{s^{p+3}}{p+3} \bigl\langle \underset{1}{f}{}_p \bigr\rangle + 4\pi \ln\left(\frac{s}{s_1}\right) \bigl\langle \underset{1}{f}{}_{-3} \bigr\rangle + 1 \leftrightarrow 2 \biggr\}. \end{aligned} \tag{3.4}$$

The description of this formula in words is as follows. One first excises two *spherical* balls $B_1(s)$ and $B_2(s)$ surrounding the two singularities (each one having the same radius s), and one computes the integral on the volume external to these balls, *i.e.*, $\mathbb{R}^3 \setminus B_1(s) \cup B_2(s)$ — first term in Eq. (3.4). That integral tends to infinity when $s \to 0$, but we can subtract from it its purely divergent part, which is given by the additional terms in (3.4) (which obviously are to be duplicated when there are 2 singularities, *cf.* the symbol $1 \leftrightarrow 2$). The limit $s \to 0$ then exists (by definition) and defines Hadamard's partie finie.

Notice the crucial dependence of the partie finie on two constants s_1 and s_2 entering the log-terms. These constants have the dimension of length. We shall say that s_1 is the regularization length scale associated to the Hadamard regularization of the divergencies near $\mathbf{x} = \mathbf{y}_1$ (similarly for s_2). Note also that the Hadamard partie finie does not depend (modulo changing the values of s_1 and s_2) on the *shape* of the regularization volumes B_1 and B_2, above chosen as simple spherical balls (see the discussion in Ref. [23]).

An important consequence of the definition (3.4) is that, in general, the integral of a gradient $\partial_i F$ is not zero, because the surface integrals surrounding the singularities become infinite when the surface areas tend to zero, and may possess a finite part. We find (see Eq. (3.4) in [23])

$$\mathrm{Pf} \int d^3\mathbf{x}\, \partial_i F(\mathbf{x}) = -4\pi \bigl\langle n_1^i \underset{1}{f}{}_{-2} \bigr\rangle + 1 \leftrightarrow 2\,. \tag{3.5}$$

For a general $F \in \mathcal{F}$ the R.H.S. is typically non-zero. This fact shows that the application of the ordinary Hadamard regularization in the post-Newtonian iteration has to be supplemented by a notion of distributional derivatives, in order to ensure that the integrals of gradients are zero like in the case of regular functions. Notice that the constants s_1 and s_2 disappear from the result (3.5). [We shall also see the need, within dim. reg., to consider some derivatives in the sense of distribution theory.]

Let us apply the definition (3.4) to the integral of a compact-support or "contact" term, *i.e.*, made of the product of some F and a Dirac delta-function at the point 1. Let us

formally assume that [11]

$$\mathrm{Pf}\int d^3\mathbf{x}\,F(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)=(F)_1\,,\tag{3.6}$$

which is the most natural way, within Hadamard's regularization, to give a sense to such integral. Now the problem with that definition is that if we want to dispose of some *local* meaning (at any field point $\mathbf{x}$) for the product of F with the delta-function, then as a consequence of the non-distributivity we cannot simply equate $F(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$ with $(F)_1\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$, *i.e.*:

$$F(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)\neq(F)_1\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)\ \text{ in general for } F\in\mathcal{F}.\tag{3.7}$$

Indeed, if it were true that $F\delta_1^{(3)}=(F)_1\delta_1^{(3)}$ [for simplicity we denote $\delta_1^{(3)}\equiv\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$], then multiplying by any G we would have $FG\delta_1^{(3)}=(F)_1G\delta_1^{(3)}$, and by integrating over $\mathbb{R}^3$ following the rule (3.6) this would yield $(FG)_1=(F)_1(G)_1$ in contradiction with the violation of distributivity (3.3). Therefore, both the violation of distributivity (3.3) and the consequence (3.7) are unescapable in the ordinary Hadamard regularization.

The previous situation should be contrasted with the d-dimensional case for which the distributivity is always satisfied, as we have simply

$$(F^{(d)}\,G^{(d)})(\mathbf{y}_1)=F^{(d)}(\mathbf{y}_1)\,G^{(d)}(\mathbf{y}_1)\,,\tag{3.8}$$

and

$$F^{(d)}(\mathbf{x})\,\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)=F^{(d)}(\mathbf{y}_1)\,\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)\,.\tag{3.9}$$

Finally, taking the $d\to3$ limit, we see that the dim. reg. way of regularizing a three-dimensional "contact term", *i.e.* a term like $F(\mathbf{x})\,\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$, is by considering it as the $d\to3$ limit of its d-dimensional analogue (3.9). That is,

$$\text{dim. reg.}\,\big[F(\mathbf{x})\,\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)\big]\equiv\Big(\lim_{d\to3}F^{(d)}(\mathbf{y}_1)\Big)\,\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)\,,\tag{3.10}$$

where $F^{(d)}$ is the d-dimensional version of F, as obtained by solving Einstein's equations in d dimensions (using the method explained in Section IV A below). There are no poles in the calculation of the "contact" terms in any of the potentials at the 3PN order so the limit $d\to3$ in Eq. (3.10) always exists. Once again the dim. reg. prescription (3.10) owns all the good features one wishes, notably the distributivity as we have emphasized in Eqs. (3.8)-(3.9).

In the following it will be convenient, in order to compare the present dim.-reg. calculation with the Hadamard-based work [22], to introduce the terminology *pure Hadamard* regularization to refer to the following "minimal" version of the Hadamard regularization: (a) an integral $\int d^3\mathbf{x}\,F(\mathbf{x})$, where F is made of some product of derivatives of the non-linear potentials V, V_i, $\cdots$, is regularized by the ordinary Hadamard partie finie prescription (3.4), without bringing in any distributional contributions (see below for the treatment of these);

[11] Actually this assumption should be viewed as the *definition* of a new object we can call $\mathrm{Pf}\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$ and which takes the property (3.6). This is exactly what we do in the context of the extended Hadamard regularization.

(b) the regularization of a product of potentials V, V_i, $\hat{W}_{ij}$, $\cdots$ (and their gradients) is assumed to be distributive, which means that the value at the singular point $\mathbf{y}_1$ of some polynomial in V, V_i, $\hat{W}_{ij}$, $\cdots$ and their gradients, say $\mathcal{F}[V, V_i, \hat{W}_{ij}, \partial_i V, \cdots]$, is given by the replacement rule

$$\left(\mathcal{F}[V, V_i, \hat{W}_{ij}, \partial_i V, \cdots]\right)_1 \longrightarrow \mathcal{F}[(V)_1, (V_i)_1, (\hat{W}_{ij})_1, (\partial_i V)_1, \cdots]\,; \tag{3.11}$$

and (c) a contact term, *i.e.* of the form $F(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$, appearing in the calculation of the *sources* of the non-linear potentials, is regularized by using the rule

$$\mathcal{F}[V, V_i, \hat{W}_{ij}, \cdots]\delta^{(3)}(\mathbf{x}-\mathbf{y}_1) \longrightarrow \mathcal{F}[(V)_1, (V_i)_1, (\hat{W}_{ij})_1, \cdots]\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)\,, \tag{3.12}$$

(there are no gradients of potentials in the contact terms). The rules (3.11)-(3.12) of the pure-Hadamard regularization are formally equivalent to assuming the replacement rules $(FG)_1 \longrightarrow (F)_1(G)_1$ together with $F(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}_1) \longrightarrow (F)_1\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$, in the case where F and G are made of products of our elementary potentials and their gradients.[12] The rules of the pure Hadamard regularization are, however, well defined, and are not submitted (by their very definition) to the consequences of the *ordinary* Hadamard regularization (3.3) and (3.7). Note also that, as done in previous computations of the 3PN ADM-Hamiltonian [16, 17] and the 3PN binary's energy flux [27], one can formally use (3.11)-(3.12) at the price of adding a limited number of arbitrary parameters (considered as unknown).

The definition (3.12) of pure-Hadamard regularization for contact terms is useful because we have checked that, when using the dim. reg. prescription (3.10) (in the limit where $d \to 3$), all the contact terms in the sources of the non-linear potentials V, V_i, $\hat{W}_{ij}$, $\cdots$ needed at the 3PN order, *agree* with the result of the pure-Hadamard regularization. [Of course, we would not need to introduce a notion of pure-Hadamard regularization in a direct calculation of the equations of motion in d dimensions, *i.e.*, not based on the "difference" between Hadamard and dim. reg., because in such a pure dim. reg. approach the contact terms would be treated unambiguously from the start using Eq. (3.10).] On the other hand, when computing the value at the singular point of the potentials for insertion into the geodesic equations, we do find some departure between the dim.-reg. calculation and the (ordinary or extended) Hadamard one. Let us illustrate these differences by means of the simplest example which does enter our 3PN calculation, namely the regularization of $(U)^3\partial_i U$ where U is the Newtonian potential. In d dimensions $U^{(d)}(\mathbf{x})$ is given by Eq. (2.21) [we add here a superscript (d) to indicate the d-dimensionality of a potential and pose $U \equiv U^{(3)}$]. Therefore in dim. reg. the result is simply

$$\lim_{d\to 3}\Big([U^{(d)}(\mathbf{y}_1)]^3\partial_i U^{(d)}(\mathbf{y}_1)\Big) = [U_2(\mathbf{y}_1)]^3\partial_i U_2(\mathbf{y}_1) \quad \text{(dim. reg.)}\,, \tag{3.13}$$

where $U_2(\mathbf{y}_1) = Gm_2/r_{12}$ is the value at point 1 of the potential of the other particle. The result (3.13) is the same as when using the pure Hadamard regularization. Indeed, we find first that $(U)_1 = U_2(\mathbf{y}_1)$ and $(\partial_i U)_1 = \partial_i U_2(\mathbf{y}_1)$, and then, by using the definition (3.11),

$$(U^3\partial_i U)_1 \xrightarrow{\text{def}} [(U)_1]^3(\partial_i U)_1 = [U_2(\mathbf{y}_1)]^3\partial_i U_2(\mathbf{y}_1) \quad \text{(pure-Hadamard)}\,. \tag{3.14}$$

[12] Thus we shall write $(V\hat{W}_{ij})_1 \longrightarrow (V)_1(\hat{W}_{ij})_1$ or $(V^3\partial_i V)_1 \longrightarrow [(V)_1]^3(\partial_i V)_1$, but not, for instance, $(V)_1 \longrightarrow [(\sqrt{V})_1]^2$.

On the other hand, the latter results contrast with the application of the ordinary Hadamard regularization for which we find

$$(U^3\partial_i U)_1 = [U_2(\mathbf{y}_1)]^3\partial_i U_2(\mathbf{y}_1) + \frac{6}{5}[U_1(\mathbf{y}_2)]^2 U_2(\mathbf{y}_1)\partial_i U_2(\mathbf{y}_1) \quad \text{(ordinary-Hadamard)}\,. \tag{3.15}$$

The first term is in fact the "pure-Hadamard" result which is in agreement with the dim. reg. one. The second term is an example of the non-distributivity of the ordinary Hadamard regularization,[13] which is also systematically taken into account in the extended-Hadamard regularization that we shall describe in Section III C.

B. Ordinary Hadamard regularization of three-dimensional Poisson integrals

Let us give some reminders of the way we apply the considerations of Section III A to the computation of Hadamard-regularized potentials having the form of Poisson or Poisson-like integrals. Let us first discuss the prescription one has taken in $d = 3$ to define the "value at $\mathbf{x}' = \mathbf{y}_1$" of a (singular) Poisson potential $P(\mathbf{x}')$. In $d = 3$, the Poisson integral $P(\mathbf{x}')$, at some field point $\mathbf{x}' \in \mathbb{R}^3$, of some singular source function $F(\mathbf{x})$ in the class $\mathcal{F}$ is defined in the sense of the partie-finie integral (3.4), namely

$$P(\mathbf{x}') = -\frac{1}{4\pi}\mathrm{Pf}_{s_1,s_2}\int \frac{d^3\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|}F(\mathbf{x})\,, \tag{3.16}$$

where s_1 and s_2 are the two constants introduced in Eq. (3.4). At first sight we could think that a good choice for defining the pure Hadamard value $[P(\mathbf{x}')]_{\mathbf{x}'=\mathbf{y}_1}$ is simply to replace $\mathbf{x}' = \mathbf{y}_1$ in (3.16), *i.e.*,

$$P(\mathbf{y}_1) \equiv -\frac{1}{4\pi}\mathrm{Pf}_{s_1,s_2}\int \frac{d^3\mathbf{x}}{r_1}F(\mathbf{x})\,. \tag{3.17}$$

However, the work on the 3PN equations of motion [21, 22] suggested that the definition (3.17) is not acceptable: it did not seem to be able to yield equations of motion compatible with basic physical properties such as energy conservation.

The choice adopted in [21, 22] is to define the regularized "value at $\mathbf{x}' = \mathbf{y}_1$" of the function $P(\mathbf{x}')$ by taking the Hadamard partie finie in the singular limit $\mathbf{x}' \to \mathbf{y}_1$. Notice first that $P(\mathbf{x}')$ does not belong (in general) to the class $\mathcal{F}$ because the Poisson integral will generate some *logarithms* of r'_1 in its expansion when $r'_1 \to 0$. Thus, we shall have, rather than an expansion of type (3.1),

$$P(\mathbf{x}') = \sum_{p_0\leq p\leq N} {r'_1}^p\left[\underset{1}{g}_p(\mathbf{n}'_1) + \underset{1}{h}_p(\mathbf{n}'_1)\ln r'_1\right] + o({r'_1}^N)\,, \tag{3.18}$$

[13] In the ADM-Hamiltonian the analogue of this example is the regularization of U^4, which gives automatically $[U_2(\mathbf{y}_1)]^4$ in dim. reg. and also (by definition) in the pure-Hadamard reg., while

$$(U^4)_1 = [U_2(\mathbf{y}_1)]^4 + 2[U_1(\mathbf{y}_2)]^2[U_2(\mathbf{y}_1)]^2\,.$$

The latter example represents in fact the only source of ambiguity present in (the static part of) the ADM-Hamiltonian formalism [35].

where the coefficients ${}_1g_p$ and ${}_1h_p$ depend on the angles $\mathbf{n}'_1$, and also on the constants s_1 and s_2, in such a way that when combining together the terms in (3.18) the constant r'_1 always appears in "adimensionalized" form like in $\ln(r'_1/s_1)$. Then we define the Hadamard partie finie at point 1 exactly in the same way as in Eq. (3.2), except that we now include a contribution linked to the (divergent) logarithm of r'_1, which is possibly present into the zeroth-order power of r'_1. More precisely, we define

$$(P)_1 \equiv \langle \underset{1}{g}{}_0 \rangle + \langle \underset{1}{h}{}_0 \rangle \ln r'_1 \,, \tag{3.19}$$

where we introduced a *new regularization length scale* denoted r'_1, which can be seen as some "small" but finite cut-off length scale [so that $\ln r'_1$ in Eq. (3.19) is a finite, but "large" cut-off dependent contribution]. We shall see later that the dependence on r'_1 disappears (as it should) when adding to $(P)_1$ the difference $\mathcal{D}P(1) \equiv P^{(d)}(\mathbf{y}_1) - (P)_1$. To compute the partie finie one must apply the definition (3.19) to the Poisson integral (3.16), which involves evaluating correctly the angular integration therein. The result, proved in Theorem 3 of [23], is

$$(P)_1 = -\frac{1}{4\pi}\mathrm{Pf}_{s_1,s_2}\int \frac{d^3\mathbf{x}}{r_1}F(\mathbf{x}) + \left[\ln\left(\frac{r'_1}{s_1}\right) - 1\right]\langle \underset{1}{f}{}_{-2} \rangle \,. \tag{3.20}$$

We recover in the first term the value of the potential at the point 1: $P(\mathbf{y}_1)$, given by Eq. (3.17). The supplementary term makes the partie finie to differ from the "naïve" guess $P(\mathbf{y}_1)$ in a way which was found to play a significant role in the computations of [21, 22]. The apparent dependence of the result (3.20) on the scale s_1 is illusory. The s_1-dependence of the R.H.S. of Eq. (3.20) cancels between the first and the second term, so the result depends only on the constants r'_1 and s_2, and we have in fact the following simpler rewriting of (3.20),

$$(P)_1 \equiv -\frac{1}{4\pi}\mathrm{Pf}_{r'_1,s_2}\int \frac{d^3\mathbf{x}}{r_1}F(\mathbf{x}) - \langle \underset{1}{f}{}_{-2} \rangle \,. \tag{3.21}$$

Similarly the regularization performed at point 2 will depend on r'_2 and s_1, so that the binary's point-particle dynamics in Hadamard's regularization depends on four (*a priori* independent) length scales r'_1, s_2 and r'_2, s_1. The explicit expression of the result (3.21) is readily obtained from the definition of the partie-finie integral (3.4). We find (see the details in Ref. [23])

$$\begin{aligned}(P)_1 = \lim_{s\to 0}\Bigg\{ & -\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B_1(s)\cup B_2(s)} \frac{d^3\mathbf{x}}{r_1}F(\mathbf{x}) \\ & - \sum_{p+2<0}\frac{s^{p+2}}{p+2}\langle \underset{1}{f}{}_p \rangle - \left[\ln\left(\frac{s}{r'_1}\right) + 1\right]\langle \underset{1}{f}{}_{-2} \rangle \\ & - \sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}}\right)\left[\sum_{p+\ell+3<0}\frac{s^{p+\ell+3}}{p+\ell+3}\langle n_2^L \underset{2}{f}{}_p \rangle + \ln\left(\frac{s}{s_2}\right)\langle n_2^L \underset{2}{f}{}_{-\ell-3} \rangle\right]\Bigg\}.\end{aligned} \tag{3.22}$$

Note that the terms corresponding to singularity 2 involve the multipolar expansion around

the point $\mathbf{y}_2$ of the factor $1/r_1 = 1/|\mathbf{x}-\mathbf{y}_1|$ present into the integrand.[14]

Because we work at the level of the equations of motion, many of the terms we shall need in this paper are in the form of the *gradient* of a Poisson-like potential. For the gradient we have a formula analogous to (3.21) and given by Eq. (5.17a) of [23], namely

$$\begin{aligned}(\partial_i P)_1 &= -\frac{1}{4\pi}\mathrm{Pf}_{s_1,s_2}\int d^3\mathbf{x}\frac{n_1^i}{r_1^2}F(\mathbf{x}) + \ln\left(\frac{r_1'}{s_1}\right)\langle n_1^i \underset{1}{f}_{-1}\rangle \\ &= -\frac{1}{4\pi}\mathrm{Pf}_{r_1',s_2}\int d^3\mathbf{x}\frac{n_1^i}{r_1^2}F(\mathbf{x})\,, \end{aligned} \tag{3.23}$$

where we have taken into account (in the rewriting of the second line) the always correct fact that the constant s_1 cancels out and gets "replaced" by r_1'. Notice that in (3.23) there is no additional term to the partie finie integral similar to the last term in (3.21). The corresponding explicit expression is

$$\begin{aligned}(\partial_i P)_1 = \lim_{s\to 0}\Bigg\{ &-\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B_1(s)\cup B_2(s)} d^3\mathbf{x}\,\frac{n_1^i}{r_1^2}F(\mathbf{x}) \\ &-\sum_{p+1<0}\frac{s^{p+1}}{p+1}\langle n_1^i \underset{1}{f}_{p}\rangle - \ln\left(\frac{s}{r_1'}\right)\langle n_1^i \underset{1}{f}_{-1}\rangle \\ &-\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_{iL}\left(\frac{1}{r_{12}}\right)\left[\sum_{p+\ell+3<0}\frac{s^{p+\ell+3}}{p+\ell+3}\langle n_2^L \underset{2}{f}_{p}\rangle + \ln\left(\frac{s}{s_2}\right)\langle n_2^L \underset{2}{f}_{-\ell-3}\rangle\right]\Bigg\}. \end{aligned} \tag{3.24}$$

Finally we must also treat the more general case of potentials in the form of retarded integrals [see Eqs. (2.12)], but because we shall have to consider (in Section IV B) only the *difference* between the dimensional and Hadamard regularizations, it will turn out that in fact the first-order retardation (1PN relative order) is sufficient for this purpose. Actually, in this paper we are not interested in radiation-reaction effects, so we shall use the symmetric (half-retarded plus half-advanced) integral. At the 1PN order we thus have to evaluate

$$R(\mathbf{x}') = P(\mathbf{x}') + \frac{1}{2c^2}Q(\mathbf{x}') + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{3.25}$$

where $P(\mathbf{x}')$ is given by (3.16), and where $Q(\mathbf{x}')$ denotes (two times) the double or "twice-iterated" Poisson integral of the second-time derivative, still endowed with a prescription of

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[14] We write the multipole expansion in the form

$$\frac{1}{r_1} = \sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}}\right)r_2^\ell n_2^L\,,$$

employing our usual notation where capital letters denote multi-indices: $L\equiv i_1 i_2\cdots i_\ell$, and, for instance, $n_2^L \equiv n_2^{i_1}\cdots n_2^{i_\ell}$. The expansion is symmetric-trace-free (STF) because $\delta_{i_\ell i_{\ell-1}}\partial_L(1/r_{12}) = \partial_{L-2}\Delta(1/r_{12}) = 0$. Here $\partial_L(1/r_{12})$ is a short-hand for ℓ partial derivatives $\partial/\partial y_{12}^i$ of $1/r_{12}\equiv 1/|\mathbf{y}_1-\mathbf{y}_2|$. The multipole expansion in d dimensions (also STF) is given by Eq. (4.23) below.

taking the Hadamard partie finie, namely

$$Q(\mathbf{x}') = -\frac{1}{4\pi}\mathrm{Pf}_{s_1,s_2}\int d^3\mathbf{x}\,|\mathbf{x}-\mathbf{x}'|\partial_t^2 F(\mathbf{x})\,. \tag{3.26}$$

In the case of $Q(\mathbf{x}')$ the results concerning the partie finie at point 1 were given by Eqs. (5.16) and (5.17b) of [23],

$$(Q)_1 \;=\; -\frac{1}{4\pi}\mathrm{Pf}_{r_1',s_2}\int d^3\mathbf{x}\,r_1\partial_t^2 F(\mathbf{x}) + \frac{1}{2}\langle \underset{1}{k}_{-4}\rangle\,, \tag{3.27a}$$

$$(\partial_i Q)_1 \;=\; \frac{1}{4\pi}\mathrm{Pf}_{r_1',s_2}\int d^3\mathbf{x}\,n_1^i\partial_t^2 F(\mathbf{x}) + \frac{1}{2}\langle n_1^i\,\underset{1}{k}_{-3}\rangle\,, \tag{3.27b}$$

where the ${}_1k_p$'s denote the analogues of the coefficients ${}_1f_p$, parametrizing the expansion of F when $r_1'\to 0$, but corresponding to the double time-derivative $\partial_t^2 F$ instead of F. [In the following we shall not need the explicit forms of the results (3.27).]

Let us clarify an important point concerning the treatment of the repeated time derivative $\partial_t^2 F(\mathbf{x})$ in Eqs. (3.27). As we are talking here about Hadamard-regularized integrals (which excise small balls around both $\mathbf{y}_1$ and $\mathbf{y}_2$), the value of $\partial_t^2 F(\mathbf{x})$ can be simply taken in the sense of ordinary functions, *i.e.*, without including eventual "distributional" contributions proportional to $\delta(\mathbf{x}-\mathbf{y}_1)$ or $\delta(\mathbf{x}-\mathbf{y}_2)$ and their derivatives. However, we know that such terms are necessary for the consistency of the calculation. This is why we must also include somewhere in our formalism the *difference* between the evaluation of these distributional terms in d dimensions, and the specific distributional contributions issued from the generalized framework used in [22]. This difference will be included in Section III D below, among the complete list of additional contributions specifically related to the use of the extended regularization approach we shall now describe.

C. Extended-Hadamard regularization

The "*extended-Hadamard*" regularization, proposed in Refs. [23, 24], tackles the particular properties of the ordinary Hadamard regularization, notably the non-distributivity of Eqs. (3.3) and (3.7), and the fact that the integral of a gradient is not zero [Eq. (3.5)]. These properties are implemented within a theory of pseudo-functions, *viz* linear forms defined on the set of singular functions $\mathcal{F}$. The use of pseudo-functions in this context enables one to give a precise meaning to the object $F\delta(\mathbf{x}-\mathbf{y}_1)$ needed in the computation of the contact terms, and which is otherwise ill-defined in distribution theory. Furthermore the use of some generalized versions of distributional derivatives permits a systematic treatment of integrals and a natural implementation of the property that the integral of a gradient is always zero. In this paper we shall content ourselves with recalling the principle of the extended-Hadamard regularization, and with presenting its "ready-to-use" consequences.

To any $F\in\mathcal{F}$ we associate the "partie finie" pseudo-function $\mathrm{Pf}F$, which is the linear form on $\mathcal{F}$ defined by the duality bracket

$$<\mathrm{Pf}F, G>\equiv \mathrm{Pf}\int d^3\mathbf{x}\,F(\mathbf{x})G(\mathbf{x})\,, \tag{3.28}$$

which means that the action of $\mathrm{Pf}F$ on any $G\in\mathcal{F}$ is the partie-finie integral, as given by (3.4), of the ordinary product. The pseudo-function $\mathrm{Pf}F$ reduces to a distribution in the

ordinary sense of Schwartz [36] when restricted to the usual set $\mathcal{D}$ of smooth functions with compact support on $\mathbb{R}^3$. The product of pseudo-functions coincides, by definition, with the ordinary point-wise product, namely $\mathrm{Pf}F.\mathrm{Pf}G \equiv \mathrm{Pf}(FG)$. In the class of pseudo-functions constructed in Ref. [23], the "Dirac-delta" pseudo-function $\mathrm{Pf}\delta_1$ is defined by

$$< \mathrm{Pf}\delta_1, F >\equiv \mathrm{Pf}\int d^3\mathbf{x}\ \delta_1(\mathbf{x})F(\mathbf{x}) \equiv (F)_1\ , \tag{3.29}$$

where $(F)_1$ denotes Hadamard's partie finie (3.2). This definition, which obviously yields a natural extension of the Dirac function $\delta_1(\mathbf{x}) \equiv \delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$ in the context of Hadamard's regularization, leads also to new objects which have no equivalent in distribution theory, the most important one being the pseudo-function $\mathrm{Pf}(F\delta_1)$ which played a crucial role in [21, 22] for the calculation of the compact-support parts of potentials as well as the purely distributional parts of derivatives. It is given by

$$< \mathrm{Pf}(F\delta_1), G >\equiv (FG)_1\ , \tag{3.30}$$

where one should be reminded that it is in general not allowed to replace the R.H.S. by the product of regularizations: $(FG)_1 \neq (F)_1(G)_1$.

In the actual computation [21, 22] the pseudo-function $\mathrm{Pf}(F\delta_1)$ acts always on smooth functions with compact support ($\in \mathcal{D}$), in which case it reduces to a distribution in the ordinary sense, which was shown to admit the "intrinsic" form

$$\mathrm{Pf}(F\delta_1) = \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!}\langle n_1^L \underset{1}{f}_{-\ell}\rangle \partial_L\delta_1 \quad \text{(when restricted to } \mathcal{D}\text{)}\,. \tag{3.31}$$

Here $L \equiv i_1\cdots i_\ell$ denotes a multi-index composed of ℓ multipolar indices $i_1,\cdots,i_\ell$, $\partial_L \equiv \partial_{i_1}\cdots\partial_{i_\ell}$ means a product of ℓ partial derivatives $\partial_i = \partial/\partial x^i$, and $n_1^L \equiv n_1^{i_1}\cdots n_1^{i_\ell}$ a product of ℓ unit vectors (we do not write the ℓ summation symbols, from 1 to 3, over the indices composing L). Notice that the sum in Eq. (3.31) is finite because F admits some maximal order of divergency when $r_1 \to 0$. Now we discover that the "monopole" term in the latter multipolar sum, having $\ell = 0$, is nothing but $(F)_1\delta_1$ which is exactly the result we would get following the pure-Hadamard regularization rule (3.12). [Indeed, as we are considering here only the contact terms entering the source terms for the 3PN-level non-linear potentials, the "ordinary" Hadamard regularization $(F)_1$ coincides with the "pure" Hadamard regularization (3.12).] The sum of the other terms then define what we can call some non-distributive contributions because their appearance is the direct consequence of the violation of distributivity (3.3). Thus,

$$\mathrm{Pf}(F\delta_1) = (F)_1\delta_1\ +\ \text{"non-distributivity" contributions}\,. \tag{3.32}$$

In the work [22] care has been taken of all such non-distributivity terms. Consider for instance the Poisson integral of a compact-support term $\mathrm{Pf}(F\delta_1)$ (say, proportional to the matter source densities σ, σ_i or σ_{ij}). Using (3.31) the Poisson integral reads [15]

$$\int \frac{d^3\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|}\mathrm{Pf}(F\delta_1) = \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!}\langle n_1^L \underset{1}{f}_{-\ell}\rangle \partial_L\left(\frac{1}{r'_1}\right)\ , \tag{3.33}$$

[15] To apply (3.31) we assume that $\mathbf{x}'$ is distinct from the 2 singularities $\mathbf{y}_1$ and $\mathbf{y}_2$; see [23] for more details.

[where $\partial_L\,(1/r'_1)$ should be better written $\partial'_L\,(1/r'_1)$]. Evaluating now the partie finie (3.2) at both singular points [*i.e.*, when $r'_1 \to 0$ and $r'_2 \to 0$] we obtain

$$\left(\int \frac{d^3\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|}\mathrm{Pf}(F\delta_1)\right)_1 \;=\; 0\,, \tag{3.34a}$$

$$\left(\int \frac{d^3\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|}\mathrm{Pf}(F\delta_1)\right)_2 \;=\; \sum_{\ell=0}^{+\infty}\frac{1}{\ell!}\langle n_1^L \underset{1}{f}_{-\ell}\rangle\partial_L\left(\frac{1}{r_{12}}\right) = \int \frac{d^3\mathbf{x}}{r_2}\mathrm{Pf}(F\delta_1). \tag{3.34b}$$

The result (3.34a) is in agreement with the pure-Hadamard regularization; however Eq. (3.34b) does involve some extra terms with respect to the pure-Hadamard calculation, since the latter is easily seen to simply yield $(F)_1/r_{12}$, which is nothing but the "monopolar" term $\ell=0$ of the multipolar sum in the R.H.S. of (3.34b). Therefore we decompose (3.34b) as

$$\left(\mathrm{Pf}\int \frac{d^3\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|}F\delta_1\right)_2 = \frac{(F)_1}{r_{12}} \;+\; \text{"non-distributivity" contributions}\,. \tag{3.35}$$

The second ingredient of the extended-Hadamard regularization concerns the treatment of partial derivatives in some (extended) distributional sense. Essentially, one requires [23] that the derivative reduces to the ordinary derivative in the case of regular functions, and is such that one can integrate by parts any integrals. The latter property (valid for the spatial derivative) translates into

$$<\partial_i(\mathrm{Pf}F), G>= -<\partial_i(\mathrm{Pf}G), F>\;. \tag{3.36}$$

This rule contains the standard definition of the distributional derivative [36] as a particular case. It implies the important property that the integral of a divergence is zero. Let us pose

$$\partial_i(\mathrm{Pf}F) = \mathrm{Pf}(\partial_i F) + \mathrm{D}_i[F]\,, \tag{3.37}$$

where $\mathrm{Pf}(\partial_i F)$ denotes the derivative of F viewed as an "ordinary" pseudo-function, and $\mathrm{D}_i[F]$ represents the purely distributional part of the spatial derivative (with support concentrated on $\mathbf{y}_1$ or $\mathbf{y}_2$).

Looking for explicit solutions of the basic relation (3.36) we have found [23], with the help of Eq. (3.5):

$$\mathrm{D}_i[F] = 4\pi\,\mathrm{Pf}\left(n_1^i\left[\frac{1}{2}r_1\underset{1}{f}_{-1} + \sum_{k\geq 0}\frac{1}{r_1^k}\underset{1}{f}_{-2-k}\right]\delta_1\right) + 1 \leftrightarrow 2\,. \tag{3.38}$$

Notice that $\mathrm{D}_i[F]$ depends only on the *singular* coefficients of F (coefficients of negative powers of r_1 in the expansion of F). The derivative operator defined by Eqs. (3.37)-(3.38) does not represent the unique solution of (3.36), but it has been checked during the calculation [22] that using another possible solution results in *physically equivalent* equations of motion at 3PN order (*i.e.*, reducing to each other by a gauge transformation). Concerning multiple derivatives we dispose of the general formula

$$\mathrm{D}_L[F] = \sum_{k=1}^{\ell}\partial_{i_1\dots i_{k-1}}\mathrm{D}_{i_k}[\partial_{i_{k+1}\dots i_\ell}F]\,, \tag{3.39}$$

giving the distributional term associated with the ℓ-th spatial derivative, $\mathrm{D}_L[F] \equiv \partial_L \mathrm{Pf} F - \mathrm{Pf} \partial_L F$ (where $L = i_1 i_2 \cdots i_\ell$), in terms of the single derivative $\mathrm{D}_i[F]$. As an example, to treat the second-derivative of the Newtonian potential, $\partial_{ij} U$ where $U = G m_1/r_1 + G m_2/r_2$, one uses

$$\mathrm{D}_{ij}\left[\frac{1}{r_1}\right] = -\frac{4\pi}{3}\,\mathrm{Pf}\left(\delta^{ij} + \frac{15}{2}\hat{n}_1^{ij}\right)\delta_1\ , \tag{3.40}$$

where $\hat{n}_1^{ij} \equiv n_1^{ij} - \frac{1}{3}\delta_{ij}$. Therefore the extended distributional derivative differs in general from the usual Schwartz's derivative [*cf.* the second term in (3.40)]. [This is unavoidable if one wants to respect the basic rule of integration by parts (3.36) for general functions in the class $\mathcal{F}$.] Notice also that we do find a distributional term in the case of the first derivative: $\mathrm{D}_i\left[1/r_1\right] = 2\pi\,\mathrm{Pf}\left(r_1 n_1^i \delta_1\right)$. We recall also (for future use) the case of the partial time-derivative, $\partial_t(\mathrm{Pf}F) = \mathrm{Pf}(\partial_t F) + \mathrm{D}_t[F]$, whose distributional term is given by (following Ref. [23])

$$\mathrm{D}_t[F] = v_1^i \underset{1}{\mathrm{D}}{}_i[F] + v_2^i \underset{2}{\mathrm{D}}{}_i[F]\,, \tag{3.41}$$

in terms of the partial derivatives with respect to the *source* points $\mathbf{y}_1$ and $\mathbf{y}_2$, namely ${}_1\mathrm{D}_i[F]$ and ${}_2\mathrm{D}_i[F]$. The explicit expression reads

$$\mathrm{D}_t[F] = -4\pi\,\mathrm{Pf}\left((n_1 v_1)\left[\frac{1}{2} r_1 \underset{1}{f}{}_{-1} + \sum_{k\geq 0} \frac{1}{r_1^k} \underset{1}{f}{}_{-2-k}\right]\delta_1\right) + 1 \leftrightarrow 2\,, \tag{3.42}$$

where $(n_1 v_1)$ denotes the ordinary scalar product [notice the over-all sign difference with respect to Eq. (3.38)]. Multiple time-derivatives can be treated accordingly to Eq. (3.39). For instance,

$$\mathrm{D}_{tt}[F] = \mathrm{D}_t[\partial_t F] + \partial_t \mathrm{D}_t[F]\,. \tag{3.43}$$

Following the regularization [23] all the distributional terms [of type $\mathrm{Pf}(F\delta_1)$ and $\mathrm{Pf}(F\delta_2)$] issued from the latter distributional derivatives are to be treated when computing the potentials according to the extended contact term definitions of Eqs. (3.34a)-(3.34b).

Finally let us turn to the extension of the Hadamard regularization (introduced in [24]) concerning the definition of a new operation of regularization, denoted $[F]_1$, consisting of performing the Hadamard regularization $(F)_1$ within the spatial hypersurface that is geometrically orthogonal (in a Minkowskian sense) to the four-velocity of the particle 1. The regularization $[F]_1$ differs from $(F)_1$ by a series of relativistic corrections calculated in [24]. Together with the other improvements of the extended-Hadamard regularization, it resulted in equations of motion in harmonic-coordinates which are manifestly Lorentz invariant at the 3PN order [21, 22]. Here we give a formula, sufficient for the present purpose, for expressing $[F]_1$ in terms of the basic regularization $(F)_1$, defined by (3.2), at the 1PN order:

$$[F]_1 = \left(F + \frac{1}{c^2}(\mathbf{r}_1.\mathbf{v}_1)\left[\partial_t F + \frac{1}{2} v_1^i \partial_i F\right]\right)_1 + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{3.44}$$

The first term is simply $(F)_1$, while the other terms define a set of relativistic corrections required for ensuring the Lorentz invariance of the final equations in Hadamard's regularization. Hence, we decompose (3.44) into

$$[F]_1 = (F)_1 \ + \ \text{“Lorentz” contributions}\,. \tag{3.45}$$

D. Contributions due to the extended-Hadamard regularization

After the reminders of the last subsections, we are now in position to explain the origin of all the contributions [included in the final result (1.5)] which were due to the specific use of the extended-Hadamard regularization. Actually, we shall list here the contributions due to the use of the full prescriptions of [23, 24] with respect to those that would follow from using what we shall call a "*pure Hadamard-Schwartz*" (pHS) regularization. By this we mean: (1) treating the contact terms of all the non-linear potentials V, V_i, $\hat{W}_{ij}$, $\cdots$ as in (3.12) [we have checked that for all the potentials involved this is equivalent to (3.10)]; (2) treating the distributional part of an integrand such as $F_{ij}\,\partial_{ij}\,U$ in the normal Schwartz's distributional way, for instance [16]

$$\partial_{ij}\left(\frac{1}{r_1}\right)=\frac{3\,n_1^i\,n_1^j-\delta^{ij}}{r_1^3}-\frac{4\pi}{3}\,\delta_{ij}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}_1)\,, \tag{3.46}$$

and evaluating the contact term generated by the delta function in the "pure Hadamard" way (3.12); (3) regularize any three-dimensional integral by the ordinary Hadamard prescription (3.4); and, finally, (4) using systematically, in the last stage of the calculation where one replaces the metric into the geodesic equations, the pure Hadamard replacement rule appearing in (3.11) [for instance, we write $(V^3\partial_i V)_1 \longrightarrow [(V)_1]^3(\partial_i V)_1$, creating therefore a net difference with respect to the ordinary and/or extended Hadamard regularizations for which $(V^3\partial_i V)_1 \neq [(V)_1]^3(\partial_i V)_1$].

The usefulness of the definition of such a pHS regularization is to "localize" the additional contributions brought by dim. reg. to the occurrence of poles $\propto 1/\varepsilon$ (or "cancelled" poles) in d dimensions.

Our complete list of additional contributions contains seven items. First of all there are four "non-distributivity" contributions of the type given by Eq. (3.35):

(i) The so-called "self" terms, for which the delta-function in $\mathrm{Pf}(F\delta_1)$ comes from the purely distributional part of the distributional derivative given by (3.38). The self terms were derived in Eq. (6.20) in [22]; they read explicitly

$$\delta^{\text{self}}a_1^i=\frac{151}{9}\frac{G^4m_1m_2^3}{c^6r_{12}^5}n_{12}^i+\frac{G^3m_2^3}{c^6}\left[-\frac{1}{2}(n_{12}v_2)^2n_{12}^i+\frac{1}{10}v_2^2n_{12}^i+\frac{1}{5}(n_{12}v_2)v_2^i\right], \tag{3.47}$$

where $(n_{12}v_2)$ denotes the usual scalar product between $\mathbf{n}_{12}$ and $\mathbf{v}_2$, and where $v_2^2=\mathbf{v}_2^2$. The expression (3.47) can be rewritten in a simpler way as (where we denote for simplicity $v_2^{jk}\equiv v_2^jv_2^k$)

$$\delta^{\text{self}}a_1^i=\frac{151}{9}\frac{G^4m_1m_2^3}{c^6r_{12}^5}n_{12}^i+\frac{1}{30}\frac{G^3m_2^3}{c^6}v_2^{jk}\partial_{ijk}\left(\frac{1}{r_{12}}\right). \tag{3.48}$$

(ii) The so-called "Leibniz" terms, which are additional contributions due to the extended distributional derivative, taking into account the violation of the Leibniz rule when performing some simplifications of the non-linear potentials at the 3PN order (see the

[16] Notice that the distributional term differs from the extended-Hadamard prescription (3.40).

explanations in Section III B in [22]). The Leibniz terms were written in Eq. (6.19) in [22], and read

$$\delta^{\mathrm{Leibniz}} a_1^i = -\frac{88}{9}\frac{G^4 m_1 m_2^3}{c^6 r_{12}^5} n_{12}^i - \frac{1}{6}\frac{G^3 m_2^3}{c^6} v_2^{jk} \partial_{ijk}\left(\frac{1}{r_{12}}\right). \tag{3.49}$$

We emphasize that the contributions (3.48) and (3.49) represent some additive effects of the use of the distributional derivative introduced in Ref. [23], when compared to the effect of the Schwartz derivative in the pHS regularization. Note that both Eqs. (3.48) and (3.49) depend on the choice of distributional derivative, and we have given them here in the case of the "particular" derivative [17] defined by (3.38).

(iii) A special non-distributivity in the compact-support potential V when it is computed at the 3PN order. In this case the $\mathrm{Pf}(F\delta_1)$ comes simply from the compact-support point-particle source σ of the potential. Eq. (4.17) of [22] gives for that term

$$\delta^{V} a_1^i = 5\frac{G^4 m_1 m_2^3}{c^6 r_{12}^5} n_{12}^i . \tag{3.50}$$

(iv) A contribution coming from the compact-support part of the potential $\hat{T}$ parametrizing the metric at the 3PN order, and derived at the end of Section IV A in [22]:

$$\delta^{\hat{T}} a_1^i = \frac{1}{15}\frac{G^3 m_2^3}{c^6} v_2^{jk} \partial_{ijk}\left(\frac{1}{r_{12}}\right). \tag{3.51}$$

Besides the non-distributivity of the type (3.35), we have also the more "direct" non-distributivity due to the fact that the pure Hadamard prescription for the regularization of the value of an expression "at $\mathbf{y}_1$", Eq. (3.11) , differs from the ordinary and/or extended Hadamard ones [see for instance Eq. (3.15)]. It plays a role only in the last stage of the computation of the 3PN equations of motion, once we substitute all the potentials computed at the right PN order into the geodesic equations. We thus have

(v) A "direct" non-distributivity contribution, which can be called non-distributivity in the equations of motion (EOM), and given by Eq. (6.34) in [22],

$$\delta^{\mathrm{EOM}} a_1^i = \frac{G^4 m_1^2 m_2}{c^6 r_{12}^5}\left[\frac{779}{210}m_1 - \frac{97}{210}m_2\right] n_{12}^i - \frac{779}{420}\frac{G^3 m_1^2 m_2}{c^6} v_{12}^{jk}\, \partial_{ijk}\left(\frac{1}{r_{12}}\right), \tag{3.52}$$

where $v_{12}^{jk} = v_{12}^j v_{12}^k$ and $v_{12}^j = v_1^j - v_2^j$. This term involves some combinations of masses different from those in Eqs. (3.48)-(3.51). Note that because $(FG)_1 \neq (F)_1(G)_1$ the non-distributivity in the EOM depends on which prescription has been chosen for the stress-energy tensor of point-particles. Eq. (3.52) corresponds to the particular prescription advocated in Section V of [24]. However it was checked in [22] that different prescriptions yield physically equivalent equations of motion.

[17] Another derivative was introduced and discussed in [23] where it is called the "correct" one, but it yields physically equivalent 3PN equations of motion.

The next correction brought about by the extended-Hadamard regularization is the one due to the regularization $[F]_1$, performed in the Lorentzian rest frame of the particle. In practice the effect of such "Lorentzian" regularization boils down to applying Eq. (3.44). It turned out that the only new contribution of this type came from the regularization of the potential $\hat{X}$ at the 1PN order [and also when deriving the result for $\delta^{\mathrm{EOM}} a_1^i$, which is Galilean-invariant, in Eq. (3.52)], leading to

(vi) The so-called "Lorentz" contribution to the acceleration, given by Eq. (5.35) in [22] as

$$\delta^{\mathrm{Lorentz}} a_1^i = \frac{G^3 m_1^2 m_2}{c^6} \left[-\frac{9}{70} v_1^{jk} + \frac{1}{5} v_1^j v_2^k \right] \partial_{ijk} \left(\frac{1}{r_{12}} \right) . \tag{3.53}$$

This term has been crucial for ensuring the Lorentz invariance of the final 3PN equations of motion in [21, 22].

Finally, one must also take care of one additional contribution (with respect to the "pHS" definitions) due to the non-Schwartzian way of treating distributional derivatives. We have already mentioned two contributions coming from this origin: (i) and (ii) above. Actually, there is a third one with the same origin and which comes from our computation (see Section IV below) of the "difference" between the dimensional and Hadamard regularizations of retarded potentials, namely the crucial potentials $\hat{X}$ and $\hat{R}_i$ which must both be expanded to 1PN fractional accuracy. More precisely, this contribution is due to the repeated time-derivative operator ∂_t^2 coming when expanding the time-symmetric Green function of the d'Alembertian as $\Box^{-1} = \Delta^{-1} + c^{-2}\Delta^{-2}\partial_t^2 + \mathcal{O}(c^{-4})$. We shall explicitly exhibit in Eqs. (4.30) below the way these derivatives enter our calculation of the difference. For technical reasons the time-derivative ∂_t^2 must be kept *inside* the integrals, so it has to be considered in a distributional sense, and we have therefore to take into account the different ways of treating the distributional derivatives in both regularizations. In the extended-Hadamard regularization the distributional terms are given by $\mathrm{D}_{tt}[F]$ which is shown in Eqs. (3.41)-(3.43), and when they enter the source of some Poisson-type integral they are evaluated according to Eqs. (3.34). On the other hand, in dim. reg. one uses the ordinary Schwartz derivative (in d dimensions) which is described in Section IV C. In this case the double time-derivative ∂_t^2 is computed with the help of the Gel'fand-Shilov formulae (4.34)-(4.35) below. When examining the difference between the contact terms in $\mathrm{D}_{tt}[F]$ and those issued from $\lim_{d\to 3} \partial_t^2 F^{(d)}$, we find that only the source for the 1PN potential $\hat{X}$ (or rather for the combination $4\partial_i \hat{X}/c^4$ which enters the equations of motion) contributes. This gives:

(vii) The following "time-derivative" contribution to the acceleration,

$$\delta^{\text{time-derivative}} a_1^i = -\frac{2}{15} \frac{G^4 m_1 m_2^3}{c^6 r_{12}^5} n_{12}^i + \frac{4}{35} \frac{G^3 m_2^3}{c^6} v_2^{jk} \, \partial_{ijk} \left(\frac{1}{r_{12}} \right) . \tag{3.54}$$

This term was part of the final result of [22]. However it is not mentioned in [22] because this reference never tried to compare the results of the extended distributional derivative with those given by the ordinary Schwartz derivative in 3 dimensions, except in those cases, *items* (i) and (ii) above, for which the Schwartz derivative yielded in fact some ill-defined (formally infinite) expressions in 3 dimensions. [The latter expressions turn out to be rigorously zero when computed in dimensional regularization.]

In summary, there are in all *seven* different terms, (i)–(vii), which are specifically due to the extended version of the Hadamard regularization. The "pure-Hadamard-Schwartz" equations of motion are then obtained from the end result of [22], *i.e.*, $\mathbf{a}_1^{\rm BF}$ given by Eq. (7.16) of [22], by subtracting these terms. Therefore we define (see also Section V B below)

$$\mathbf{a}_1^{\rm pHS} \equiv \mathbf{a}_1^{\rm BF} - \left(\delta^{\rm self}\mathbf{a}_1 + \delta^{\rm Leibniz}\mathbf{a}_1 + \delta^{V}\mathbf{a}_1 + \delta^{\hat{T}}\mathbf{a}_1 + \delta^{\rm EOM}\mathbf{a}_1 + \delta^{\rm Lorentz}\mathbf{a}_1 + \delta^{\text{time-derivative}}\mathbf{a}_1\right), \tag{3.55}$$

and *idem* with $1 \leftrightarrow 2$ for the other particle.

IV. DIMENSIONAL VERSUS HADAMARD REGULARIZATIONS

In this Section we come to the core of our technique for evaluating the difference between the d-dimensional equations of motion and their pure-Hadamard-Schwartz expressions, defined above and given in practice by Eq. (3.55).

A. Iteration of Einstein's equations in d dimensions

Let us start by indicating how we solved (with sufficient accuracy) Einstein's field equations in d dimensions. One writes the post-Minkowskian expansion of Einstein's equations in the guise of explicit formulae for the elementary potentials $V, V_i, \cdots, \hat{T}$, as given in Section II. Note that it is crucial to take into account the explicit d-dependence of the coefficients entering these equations. The first step of the formalism is to get sufficiently accurate explicit expressions for the basic linear potentials V and V_i. As we do not need to consider here radiation reaction effects (which do not mix with the UV divergencies arising at the 3PN level) it is enough to solve Eqs. (2.7) by means of the PN expansion of the time-symmetric Green function. For instance, we have

$$V = -4\pi G\,\Box^{-1}_{\rm sym}\sigma = -4\pi G\left(\Delta^{-1}\sigma + \frac{1}{c^2}\,\Delta^{-2}\,\partial_t^2\,\sigma + \frac{1}{c^4}\,\Delta^{-3}\,\partial_t^4\,\sigma + \frac{1}{c^6}\,\Delta^{-4}\,\partial_t^6\,\sigma\right) + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{4.1}$$

From Eq. (2.16) we see that the source σ reads

$$\sigma(\mathbf{x},t) = \tilde{\mu}_1(t)\,\delta^{(d)}[\mathbf{x}-\mathbf{y}_1(t)] + 1 \leftrightarrow 2\,, \tag{4.2}$$

where

$$\tilde{\mu}_1(t) = \frac{2}{d-1}\,\frac{m_1\,c}{\sqrt{-g_{\rho\sigma}(\mathbf{y}_1,t)\,v_1^\rho\,v_1^\sigma}}\,\frac{(d-2)+\mathbf{v}_1^2/c^2}{\sqrt{-g(\mathbf{y}_1,t)}}\,. \tag{4.3}$$

Note the presence of many "contact" evaluations of field quantities in σ. Such terms are unambiguously defined in dimensional regularization. They are computed by successive iterations (*e.g.* to get $\tilde{\mu}_1(t)$ to 1PN fractional accuracy we need to have already computed $g_{\mu\nu}$ to order $\mathcal{O}(c^{-2})$ included). Those evaluations do not give rise to pole terms in σ, up to the 3PN accuracy. Hence, as we said above, we can consider that their $d \to 3$ limits define a certain (three-dimensional) way of estimating contact terms, that we have checked to be in full agreement with the "pure-Hadamard" prescription defined in the previous Section.

Coming now to the spatial dependence of the scalar potential V we get from Eq. (4.1)

$$V(\mathbf{x},t) = G\,\tilde{\mu}_1(t)\,u_1 + \frac{G}{c^2}\,\partial_t^2\,[\tilde{\mu}_1(t)\,v_1] + \cdots + 1 \leftrightarrow 2\,, \tag{4.4}$$

where we introduced the following elementary solutions $u_1 \equiv \Delta^{-1}\bigl(-4\pi\delta_1^{(d)}\bigr)$, $v_1 \equiv \Delta^{-1}\, u_1$, etc., whose explicit forms are

$$u_1 = \tilde{k}\, r_1^{2-d}\,, \tag{4.5a}$$

$$v_1 = \frac{\tilde{k}\, r_1^{4-d}}{2(4-d)}\,, \tag{4.5b}$$

where $\tilde{k}$ is related to the usual Eulerian Γ-function by [18]

$$\tilde{k} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}\,. \tag{4.6}$$

Inserting the explicit expression (4.4) of V into, say, the non-linear terms in the R.H.S. of Eq. (2.12b), yields a d'Alembert equation for the non-linear potential $\hat{W}_{ij}$ with a "source function" which is the sum of some contact terms $S(\mathbf{x})\,\delta^{(d)}(\mathbf{x}-\mathbf{y}_a)$ and of an extended non-linear source $F^{(d)}(\mathbf{x})$ which belongs to the d-dimensional analogue of the class $\mathcal{F}$, say $\mathcal{F}^{(d)}$. More precisely, at each stage of the iteration we find inhomogeneous wave equations of the type

$$\Box\, W^{(d)}(\mathbf{x}) = F^{(d)}(\mathbf{x}) + \sum_a S_a(\mathbf{x})\,\delta^{(d)}(\mathbf{x}-\mathbf{y}_a)\,, \tag{4.7}$$

where the extended source function $F^{(d)}(\mathbf{x})$ is regular everywhere except at the points $\mathbf{y}_1$ and $\mathbf{y}_2$, in the vicinity of which it admits an expansion of the general form ($\forall N \in \mathbb{N}$)

$$F^{(d)}(\mathbf{x}) = \sum_{\substack{p_0\leq p\leq N\\ q_0\leq q\leq q_1}} r_1^{p+q\varepsilon} \mathop{f}_1{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1) + o(r_1^N)\,, \tag{4.8}$$

where p and q are relative integers ($p, q \in \mathbb{Z}$), whose values are limited by some p_0, q_0 and q_1 as indicated. The expansion (4.8) differs from the corresponding expansion in 3 dimensions, as given in (3.1), by the appearance of integer powers of r_1^ε where $\varepsilon \equiv d-3$. The coefficients ${}_1f^{(\varepsilon)}_{p,q}$ depend on the unit vector $\mathbf{n}_1$ in d dimensions, on the positions and coordinate velocities of the particles, and also on the characteristic length scale ℓ_0 of dimensional regularization. Because $F^{(d)} \to F$ when $d \to 3$ we necessarily have the constraint ($\forall p \geq p_0$)

$$\sum_{q_0\leq q\leq q_1} \mathop{f}_1{}^{(0)}_{p,q} = \mathop{f}_1{}_{p}\,. \tag{4.9}$$

The iteration continues by inverting the wave operator by means of the time-symmetric expansion (4.1). The basic terms of this expansion which will turn out to be crucial for our 3PN calculation based on the *difference* are in fact the first two terms. Focussing on the terms generated by the extended source $F^{(d)}(\mathbf{x})$ (rather than the simpler contact terms) we can write the d-dimensional analogue of (3.25) as

$$R^{(d)}(\mathbf{x}') \equiv \Box^{-1}_{\text{sym}}\bigl[F^{(d)}(\mathbf{x})\bigr] = P^{(d)}(\mathbf{x}') + \frac{1}{2c^2}Q^{(d)}(\mathbf{x}') + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \tag{4.10}$$

[18] The constant $\tilde{k}$ adopted here is related by $\tilde{k} = 4\pi k$ to the constant k chosen in [35]. Our present choice is motivated by the easy-to-remember fact that $\lim_{d\to 3}\tilde{k} = 1$.

where the d-dimensional Poisson integral of $F^{(d)}$ reads

$$P^{(d)}(\mathbf{x}') = \Delta^{-1}\left[F^{(d)}(\mathbf{x})\right] \equiv -\frac{\tilde{k}}{4\pi}\int \frac{d^d\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|^{d-2}} F^{(d)}(\mathbf{x})\,. \tag{4.11}$$

We have used the fact, already mentioned above, that the d-dimensional elementary solution of the Laplacian reads

$$\Delta\left(\tilde{k}|\mathbf{x}-\mathbf{x}'|^{2-d}\right) = -4\pi\delta^{(d)}(\mathbf{x}-\mathbf{x}')\,, \tag{4.12}$$

(see Appendix B for a proof of Eq. (4.12) and for other useful formulae valid in d dim.), while the 1PN term is given by

$$Q^{(d)}(\mathbf{x}') = 2\Delta^{-2}\left[\partial_t^2 F^{(d)}(\mathbf{x})\right] = -\frac{\tilde{k}}{4\pi(4-d)}\int d^d\mathbf{x}\,|\mathbf{x}-\mathbf{x}'|^{4-d}\partial_t^2 F^{(d)}(\mathbf{x})\,. \tag{4.13}$$

Note the important point that in d dimensions, as in 3 dimensions, the time-derivative operator ∂_t^2 present in the integrand of (4.13) is to be considered in the sense of distributions (see further discussion in Section IV C below).

An important technical aspect of the d-dimensional PN iteration of the elementary potentials $V, \cdots, \hat{T}$ is the existence of the generalization (4.12) of the usual Green's function for the Laplace equation, as well as of its higher PN analogues $\Delta^{-n}\,\delta^{(d)}$, allowing one to explicitly compute the spatial dependence of the *linear* potentials V, V_i and K for instance. However, starting with $\hat{W}_{ij}$ we need to Poisson-integrate *non-linear* sources, such as $\Delta^{-1}(\partial_i U\,\partial_j U)$. In three dimensions, these non-linear contributions are reducible to the knowledge of the basic non-linear potential g, such that $\Delta g = r_1^{-1}\,r_2^{-1}$. We have succeeded in explicitly computing the d-dimensional analogue of the g potential, namely

$$g^{(d)}(\mathbf{x}) \equiv \Delta^{-1}(r_1^{2-d}\,r_2^{2-d})\,. \tag{4.14}$$

Our result is reported in Appendix C. As indicated there, if we wished to explicitly compute some of the higher PN potentials needed to write the closed form of the non-linear sources relevant to the 3PN equations of motion, we should extend the calculation of the potential $g^{(d)}$ to the potentials $f^{(d)}$ and $f_{12}^{(d)}$ of Appendix C.

Luckily, it is not needed to use a closed-form expression for any of the non-linear potentials. Indeed, similarly to what was used long ago [8] when discussing the iteration generated by Riesz-type sources, Eq. (1.3), one can control the UV *singular* part of $\Box^{-1}F(\mathbf{x})$ from the knowledge of the UV singular part of its non-linear source $F(\mathbf{x})$.[19] More precisely, in the vicinity say of $\mathbf{y}_1$, at each iteration stage we can decompose the source in $F(\mathbf{x}) = \mathrm{Sing}_F(\mathbf{x}) + \mathrm{Reg}_F(\mathbf{x})$ where the *singular* part $\mathrm{Sing}_F(\mathbf{x})$ (with respect to $\mathbf{y}_1$) is a sum of terms of the form Eq. (4.8), which are not (in the limit $d\to 3$) smooth functions of $\mathbf{x}-\mathbf{y}_1$, and where the *regular* part $\mathrm{Reg}_F(\mathbf{x})$ is a smooth (C^∞) function of $\mathbf{x}-\mathbf{y}_1$. [The simplest example of this decomposition is Eq. (2.21) with, near point $\mathbf{y}_1$, $\mathrm{Sing}_U(\mathbf{x}) = f\,\tilde{k}\,G\,m_1\,r_1^{2-d}$ and $\mathrm{Reg}_U(\mathbf{x}) = U_2(\mathbf{x})$.] If, for concreteness, we then consider $P(\mathbf{x}) \equiv \Delta^{-1}F(\mathbf{x})$, the above decomposition entails a corresponding decomposition of $P(\mathbf{x})$, and it is easy to see

[19] We have checked that for all the non-compact (extended) potentials involved in this calculation, there are no IR divergencies, *i.e.*, the integrals converge at infinity $|\mathbf{x}|\to\infty$ for any small enough value of $\varepsilon = d-3$.

that $\Delta\,\mathrm{Sing}_P(\mathbf{x}) = \mathrm{Sing}_F(\mathbf{x})$. From this result, we can uniquely determine $\mathrm{Sing}_P(\mathbf{x})$ from $\mathrm{Sing}_F(\mathbf{x})$ using, *e.g.*, the formula (B26c) in Appendix B. This local procedure does not allow one to compute the regular part of the Poisson potential in d dimensions. Fortunately, thanks to particular simplifications that occur in the structure of Einstein's field equations, the knowledge of $\mathrm{Reg}_P(\mathbf{x})$ in d dimensions for the complicated non-linear sources is not needed. Indeed, one can see on our explicit solution of Einstein's field equations at 3PN given in Section II that there are no "quartically non-linear" source terms of the form, say, $\partial_i V\,\partial_j V\,\hat{W}_{ij}$ or $\partial_i \hat{W}_{jk}\,\partial_j \hat{W}_{ki}$ for g_{00} at the 3PN order (see Fig. 5 below).

As explained in Section V C below, a nice way to understand the origin of the poles $\propto (d-3)^{-1}$ appearing in the 3PN equations of motion is to use a diagrammatic representation. A pole can arise in $\mathbf{a}_1$ only when three propagator lines (including the extra one coming from $\Box^{-1}$ when solving $\Box\, g_{\mu\nu} =$ non-linear source) can all shrink towards the first world-line. If terms of the type above (*e.g.* $\partial_i V\,\partial_j V\,\hat{W}_{ij}$) were present in the source one could have a diagram where the three shrinking propagators come from $\Box^{-1}$, $\partial_i V_1$ and $\partial_j V_1$. Then $\mathrm{Reg}^{(d)}\,[\hat{W}_{ij}(\mathbf{x})]$ would remain as an external attachment to this diagram (and would then fork into two "feet" on the second world-line). In view of the pole $\propto \varepsilon^{-1}$ (with $\varepsilon \equiv d-3$) arising from the triplet of shrinking propagators, one would need to know $\mathrm{Reg}^{(d)}[\hat{W}_{ij}(\mathbf{x})]$ up to ε accuracy, *i.e.*, $\mathrm{Reg}^{(d)}[\hat{W}_{ij}(\mathbf{x})] = \mathrm{Reg}^{(3)}[\hat{W}_{ij}(\mathbf{x})] + \varepsilon\,\hat{W}'_{ij}(\mathbf{x}) + \mathcal{O}(\varepsilon^2)$ [in which $\hat{W}'_{ij}(\mathbf{x})$ is defined by this expansion]. If such a term had been present we would have needed to use the full d-dimensional, globally determined g-potential given in Appendix C to determine $\hat{W}'_{ij}$, which would have entered the final, renormalized equations of motion. However, because all such terms are absent at the 3PN order, the only external attachments to the dangerous shrinking diagrams are simple lines, such for instance as the lines ending on $\mathbf{y}_2(t)$ in Figs. 2d, 3b or 4b presented below. Such lines do need to be evaluated to accuracy ε, but this is easy because they represent linear potentials such as V or V_i which are known in dimension d *via* Eq. (4.4).

In conclusion, the algorithm we use to solve, with sufficient accuracy, Einstein's equations in d dimensions consists of: (1) starting from the fully d-dimensional expressions for the linear potentials V, V_i (and more generally for the parts of the non-linear potentials with delta-function sources); (2) determining the local expansions, near $\mathbf{y}_a$, of the singular parts of the non-linear potentials by inverting $\Delta\mathrm{Sing}_P^{(d)}(\mathbf{x}) = \mathrm{Sing}_F^{(d)}(\mathbf{x})$ *via* formulae (B26c) of Appendix B; (3) completing $P^{(d)}(\mathbf{x})$ by adding to $\mathrm{Sing}_P^{(d)}(\mathbf{x})$ the limit when $d \to 3$ of $\mathrm{Reg}_P^{(d)}(\mathbf{x})$, namely $\mathrm{Reg}_P^{(3)}(\mathbf{x})$ which is known from the previous work on the 3PN equations of motion in 3-dimensions [22]. Note that we denote by $\mathrm{Reg}_P^{(3)}(\mathbf{x})$ a formal d-dimensional function, $\mathbf{x} \in \mathbb{R}^d$, the explicit expression of which in terms of r_1, $\mathbf{n}_1$, etc. coincides with its 3-dimensional counterpart. For instance $\mathrm{Reg}_g^{(3)}(\mathbf{x})$ denotes the usual regular part of $\mathrm{g}^{(3)}(\mathbf{x})$, obtained by subtracting from $\mathrm{g}^{(3)}(\mathbf{x}) \equiv \ln(r_1 + r_2 + r_{12})$ the two three-dimensional locally singular expansions of $\Delta_{(3)}^{-1}(r_1^{-1}\,r_2^{-1})$ around $\mathbf{y}_1$ and $\mathbf{y}_2$ as given by the $d \to 3$ limit of Eq. (C9) and its $1 \leftrightarrow 2$ analog. After this double subtraction, $\mathrm{Reg}_g^{(3)}(\mathbf{x})$ is considered as a function in $\mathbb{R}^d$, and we can use as approximation to $g^{(d)}(\mathbf{x})$ the explicit expression $\mathrm{g}_{\mathrm{loc}\ 1}^{(d)}(\mathbf{x}) + \mathrm{g}_{\mathrm{loc}\ 2}^{(d)}(\mathbf{x}) + \mathrm{Reg}_g^{(3)}(\mathbf{x})$. More generally, in our calculations we use as approximation to $P^{(d)}$ [which symbolizes here the non-linear potentials $\hat{R}_i$ and $\hat{Z}_{ij}$ at Newtonian order, and $\hat{W}_{ij}$ at the 1PN order] the expression $\mathrm{Sing}_P^{(d)}(\mathbf{x}) + \mathrm{Reg}_P^{(3)}(\mathbf{x})$. Evidently, the subtraction of the singular part needs to be performed only up to some finite order in r_1^N and r_2^N. We have checked the choice we made of N in each calculation by doing two separate calculations

for the values N and $N+1$, and checking that the corresponding final results are the same. We performed also direct checks of the independence of the final results on the precise d-dimensional extensions of the "regular" part of the non-linear potentials, such as $\mathrm{Reg}_P^{(d)}(\mathbf{x}) = \mathrm{Reg}_P^{(3)}(\mathbf{x}) + \varepsilon P'(\mathbf{x}) + \mathcal{O}(\varepsilon^2)$ [in which $P'(\mathbf{x})$ is defined by this expansion]. We systematically added in all our non-linear potentials $\hat{R}_i, \cdots, \hat{W}_{ij}$ some smooth contributions to $\mathrm{Reg}_P^{(3)}(\mathbf{x})$ vanishing with ε, *i.e.*, some substitutes for the actual $P'(\mathbf{x})$. These "substitutes" were determined in such a way that (i) they are homogeneous solutions of the d'Alembertian equation at the required post-Newtonian order, (ii) the differential identities obeyed by the potentials in d dimensions, Eqs. (2.13a)-(2.13b), are indeed satisfied up to the order ε, and with the required precision N in powers[20] of r_1 or r_2. And we checked that our final results are totally insensitive to the introduction of such substitutes for the function $P'(\mathbf{x})$.

Finally, when evaluating the equations of motion, as given by Eq. (2.23), we must evaluate the value at $\mathbf{x}' = \mathbf{y}_1$ of many terms given either by Poisson integrals of the form (4.11) or their 1PN generalizations (4.13). This is quite easy to do in dim. reg., because the nice properties of analytic continuation allow simply to get $[P^{(d)}(\mathbf{x}')]_{\mathbf{x}'=\mathbf{y}_1}$ (say) by replacing $\mathbf{x}'$ by $\mathbf{y}_1$ in the explicit integral form (4.11). Finally, we simply have for the values at $\mathbf{x}' = \mathbf{y}_1$ of the potentials,

$$P^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}}{4\pi} \int \frac{d^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x})\,, \tag{4.15a}$$

$$Q^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}}{4\pi(4-d)} \int d^d\mathbf{x}\, r_1^{4-d} \partial_t^2 F^{(d)}(\mathbf{x})\,, \tag{4.15b}$$

as well as for their spatial gradients,

$$\partial_i P^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}(d-2)}{4\pi} \int d^d\mathbf{x}\, \frac{n_1^i}{r_1^{d-1}} F^{(d)}(\mathbf{x})\,, \tag{4.16a}$$

$$\partial_i Q^{(d)}(\mathbf{y}_1) = \frac{\tilde{k}}{4\pi} \int d^d\mathbf{x}\, n_1^i r_1^{3-d} \partial_t^2 F^{(d)}(\mathbf{x})\,. \tag{4.16b}$$

As said above, the main technical step of our strategy will then consist of computing the *difference* between such d-dimensional Poisson-type potentials (4.15) or (4.16), and their "pure Hadamard-Schwartz" 3-dimensional counterparts, which were already obtained in Section III B.

B. Difference between the dimensional and Hadamard regularizations

We denote the difference between the prescriptions of dimensional and "pure Hadamard-Schwartz" regularizations by means of the script letter $\mathcal{D}$. Given the results $(P)_1$ and $P^{(d)}(\mathbf{y}_1)$ of the two regularizations [respectively obtained in Eqs. (3.21) and (4.15a)] we pose

$$\mathcal{D}P(1) \equiv P^{(d)}(\mathbf{y}_1) - (P)_1\,. \tag{4.17}$$

[20] Therefore, our verification that the potentials we need do satisfy the harmonicity conditions (2.13) has been done only in the vicinity of the two particles.

That is, $\mathcal{D}P(1)$ is what we shall have to *add* to the pure Hadamard-Schwartz result (3.55) in order to get the correct d-dimensional result. Note that, in this paper, we shall only compute the first two terms, $a_{-1}\,\varepsilon^{-1} + a_0 + \mathcal{O}(\varepsilon)$, of the Laurent expansion of $\mathcal{D}P(1)$ when $\varepsilon \to 0$. This is the information we shall need to fix the value of the parameter λ. We leave to future work an eventual computation of the d-dimensional equations of motion as an exact function of the complex number d.

Similarly to the evaluation of the difference $\mathcal{D}H \equiv H^{(d)} - \text{Hadamard}[H^{(3)}]$ in Ref. [35], the difference (4.17) can be obtained by splitting the d-dimensional integral (4.15a) into three volumes, two spherical balls $B_1^{(d)}(s)$ and $B_2^{(d)}(s)$ of radius s and centered on the two singularities, and the external volume $\mathbb{R}^d \setminus B_1^{(d)}(s) \cup B_2^{(d)}(s)$. When $d \to 3$ (with fixed s), $B_1^{(d)}(s)$ and $B_2^{(d)}(s)$ tend to the regularization volumes $B_1(s)$ and $B_2(s)$ we introduced in Eq. (3.22). Consider first, for a given value $s > 0$, the external integral, over $\mathbb{R}^d \setminus B_1^{(d)}(s) \cup B_2^{(d)}(s)$. [If wished, two balls with different radii could be used, with the same result.] Since the integrand is regular on this domain, it is clear that the external integral reduces in the limit $\varepsilon \to 0$ to the one in 3 dimensions that is part of the Hadamard regularization (3.22). So we can write (for any $s > 0$)

$$-\frac{\tilde{k}}{4\pi}\int_{\mathbb{R}^d \setminus B_1^{(d)}(s)\cup B_2^{(d)}(s)} \frac{d^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3 \setminus B_1(s)\cup B_2(s)} \frac{d^3\mathbf{x}}{r_1} F(\mathbf{x}) + \mathcal{O}(\varepsilon)\,, \tag{4.18}$$

and we see that when computing the difference $\mathcal{D}P(1)$ the exterior contributions will cancel out modulo $\mathcal{O}(\varepsilon)$. Thus we obtain, after this preliminary step [following Eq. (3.22)],

$$\begin{aligned}\mathcal{D}P(1) \;=\; \lim_{s\to 0}\Bigg\{ & -\frac{\tilde{k}}{4\pi}\int_{B_1^{(d)}(s)} \frac{d^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}) - \frac{\tilde{k}}{4\pi}\int_{B_2^{(d)}(s)} \frac{d^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}) \\ & + \sum_{p\leq -3} \frac{s^{p+2}}{p+2}\langle \underset{1}{f}_{p}\rangle + \left[\ln\left(\frac{s}{r'_1}\right) + 1\right]\langle \underset{1}{f}_{-2}\rangle \\ & + \sum_{\ell\geq 0} \frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}}\right)\left[\sum_{p\leq -\ell-4} \frac{s^{p+\ell+3}}{p+\ell+3}\langle n_2^L \underset{2}{f}_{p}\rangle + \ln\left(\frac{s}{s_2}\right)\langle n_2^L \underset{2}{f}_{-\ell-3}\rangle\right]\Bigg\} \\ & + \mathcal{O}(\varepsilon)\,. \end{aligned} \tag{4.19}$$

See Section IV of Ref. [35] for a careful justification of the formal interversions of limits $s \to 0$ and $\varepsilon \to 0$ that we shall do here. The point is that in order to obtain the difference $\mathcal{D}P(1)$ we do not need the expression of $F^{(d)}$ for an arbitrary source point $\mathbf{x} \in \mathbb{R}^d$ but only in the vicinity of the two singularities: indeed the two local integrals over $B_1^{(d)}(s)$ and $B_2^{(d)}(s)$ in Eq. (4.19) can be computed by replacing $F^{(d)}$ by its expansions when $r_1 \to 0$ and $r_2 \to 0$ respectively. We substitute the r_1-expansion Eq. (4.8) into the local integral over $B_1^{(d)}(s)$, and integrate that expansion term by term. This readily leads to

$$-\frac{\tilde{k}}{4\pi}\int_{B_1^{(d)}(s)} \frac{d^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}) = -\frac{1}{1+\varepsilon}\sum_{p,q} \frac{s^{p+2+q\varepsilon}}{p+2+q\varepsilon}\langle \underset{1}{f}{}_{p,q}^{(\varepsilon)}\rangle\,, \tag{4.20}$$

where we still use the bracket notation to denote the angular average, but now performed in d dimensions, *i.e.*,

$$\langle \underset{1}{f}{}_{p,q}^{(\varepsilon)}\rangle \equiv \int \frac{d\Omega_{d-1}(\mathbf{n}_1)}{\Omega_{d-1}}\, \underset{1}{f}{}_{p,q}^{(\varepsilon)}(\mathbf{n}_1)\,. \tag{4.21}$$

Here $d\Omega_{d-1}$is the solid angle element around the direction $\mathbf{n}_1$, and $\Omega_{d-1} = 2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$ is the volume of the unit sphere with $d-1$ dimensions (see Appendix B for more discussion). To derive (4.20) we used the following relation linking $\tilde{k}$ and Ω_{d-1},

$$\tilde{k} = \frac{4\pi}{(d-2)\Omega_{d-1}} \, . \tag{4.22}$$

Concerning the other local integral, over $B_2^{(d)}(s)$, things are a little bit more involved because we need to perform a multipolar re-expansion of the factor r_1^{2-d} present in that integral around the point $\mathbf{y}_2$. Writing down this multipole expansion presents no problem, and in symmetric-trace-free (STF) form it reads [21]

$$r_1^{2-d} = \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r_{12}^{1+\varepsilon}} \right) r_2^\ell n_2^L \, . \tag{4.23}$$

The multipole expansion being then correctly taken into account, we obtain

$$-\frac{\tilde{k}}{4\pi} \int_{B_2^{(d)}(s)} \frac{d^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}) = -\frac{1}{1+\varepsilon} \sum_{p,q} \frac{s^{p+\ell+3+(q+1)\varepsilon}}{p+\ell+3+(q+1)\varepsilon} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r_{12}^{1+\varepsilon}} \right) \langle n_2^L \underset{2}{f}{}^{(\varepsilon)}_{p,q} \rangle \, . \tag{4.24}$$

As we can see, simple poles $\sim 1/\varepsilon$ will occur into our two local integrals, as determined by (4.20) and (4.24), only for the "critical" values $p = -2$ and $p = -\ell - 3$ respectively.

Next we replace the explicit expressions (4.20) and (4.24) into the formula (4.19) we had for the "difference". As expected we find that the divergencies when $s \to 0$, some value $\varepsilon \neq 0$ being given, cancel out between Eqs. (4.20)-(4.24) and the remaining terms in (4.19), so that the result is finite for any $\varepsilon \neq 0$. Furthermore, we find that if we neglect terms of order $\mathcal{O}(\varepsilon)$, the only contributions which remain are the ones coming from the poles (and their associated finite part), *i.e.*, for the latter critical values $p = -2$ in the case of singularity 1 and $p = -\ell - 3$ in the case of singularity 2. The other contributions in (4.20) and (4.24) have a finite limit when $\varepsilon \to 0$ which is therefore cancelled by the corresponding terms in Hadamard's regularization. As a result we obtain the following closed-form expression for the difference, which will constitute the basis of all the practical calculations of the present paper,

$$\begin{aligned} \mathcal{D}P(1) = &- \frac{1}{\varepsilon(1+\varepsilon)} \sum_{q_0 \leq q \leq q_1} \left(\frac{1}{q} + \varepsilon \left[\ln r_1' - 1 \right] \right) \langle \underset{1}{f}{}^{(\varepsilon)}_{-2,q} \rangle \\ &- \frac{1}{\varepsilon(1+\varepsilon)} \sum_{q_0 \leq q \leq q_1} \left(\frac{1}{q+1} + \varepsilon \ln s_2 \right) \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r_{12}^{1+\varepsilon}} \right) \langle n_2^L \underset{2}{f}{}^{(\varepsilon)}_{-\ell-3,q} \rangle \\ &+ \mathcal{O}(\varepsilon) \, . \end{aligned} \tag{4.25}$$

Notice that (4.25) depends on the two "constants" $\ln r_1'$ and $\ln s_2$. As we shall check these $\ln r_1'$ and $\ln s_2$ will exactly cancel out the same constants present in the "pure-Hadamard"

[21] The expansion is STF because $\Delta r^{2-d} = 0$ in d dimensions (in the sense of functions). See Appendix B for a compendium of d-dimensional formulae on STF expansions. See also Eq. (C6) in Appendix C.

calculation, so that the dimensionally regularized acceleration will be finally free of the constants r'_1 and s_2. Note also that the coefficients ${}_1f^{(\varepsilon)}_{p,q}$ and ${}_2f^{(\varepsilon)}_{p,q}$ in d dimensions depend on the length scale ℓ_0 associated with dimensional regularization [see Eq. (2.4)]. Taking this dependence into account one can verify that r'_1 and s_2 in (4.25) appear only in the combinations $\ln(r'_1/\ell_0)$ and $\ln(s_2/\ell_0)$.

Let us give also (without proof) the formula for the difference between the *gradients* of potentials, *i.e.*,

$$\mathcal{D}\partial_i P(1) \equiv \partial_i P^{(d)}(\mathbf{y}_1) - (\partial_i P)_1 \,. \tag{4.26}$$

The formula is readily obtained by the same method as before, and we have

$$\begin{aligned}\mathcal{D}\partial_i P(1) = &- \frac{1}{\varepsilon} \sum_{q_0\leq q\leq q_1} \left(\frac{1}{q} + \varepsilon \ln r'_1\right) \langle n_1^i \mathop{f}_{1}{}^{(\varepsilon)}_{-1,q}\rangle \\ &- \frac{1}{\varepsilon(1+\varepsilon)} \sum_{q_0\leq q\leq q_1} \left(\frac{1}{q+1} + \varepsilon \ln s_2\right) \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_{iL}\left(\frac{1}{r_{12}^{1+\varepsilon}}\right) \langle n_2^L \mathop{f}_{2}{}^{(\varepsilon)}_{-\ell-3,q}\rangle \\ &+ \mathcal{O}(\varepsilon) \,. \end{aligned} \tag{4.27}$$

Formulae (4.25) and (4.27) correspond to the difference of Poisson integrals. But we have already discussed that we shall need also the difference of inverse d'Alembertian integrals at the 1PN order. To express as simply as possible the 1PN-accurate generalizations of Eqs. (4.25) and (4.27), let us define two *functionals* $\mathcal{H}$ and $\mathcal{H}_i$ which are such that their actions on any d-dimensional function $F^{(d)}$ is given by the R.H.S.'s of Eqs. (4.25) and (4.27), *i.e.*, so that

$$\mathcal{D}P(1) \;=\; \mathcal{H}\left[F^{(d)}\right] \,, \tag{4.28a}$$

$$\mathcal{D}\partial_i P(1) \;=\; \mathcal{H}_i\left[F^{(d)}\right] \,. \tag{4.28b}$$

The difference of 1PN-retarded potentials and gradients of potentials is denoted

$$\mathcal{D}R(1) \;\equiv\; R^{(d)}(\mathbf{y}_1) - (R)_1 \,, \tag{4.29a}$$

$$\mathcal{D}\partial_i R(1) \;\equiv\; \partial_i R^{(d)}(\mathbf{y}_1) - (\partial_i R)_1 \,, \tag{4.29b}$$

where in 3 dimensions the potential $R(\mathbf{x}')$ is defined by Eq. (3.25) and the regularized values $(R)_1$ and $(\partial_i R)_1$ follow from (3.21), (3.23), (3.27), and where in d dimensions $R^{(d)}(\mathbf{y}_1)$ and $\partial_i R^{(d)}(\mathbf{y}_1)$ are given by (4.10), (4.15), (4.16). With this notation we now have our result, which will be stated without proof, that the difference in the case of such 1PN-expanded potentials reads in terms of the above defined functionals $\mathcal{H}$ and $\mathcal{H}_i$ as

$$\mathcal{D}R(1) \;=\; \mathcal{H}\left[F^{(d)} + \frac{r_1^2}{2c^2(4-d)}\partial_t^2 F^{(d)}\right] - \frac{3}{4c^2}\langle \mathop{k}_{1}{}_{-4}\rangle + \mathcal{O}\left(\frac{1}{c^4}\right) , \tag{4.30a}$$

$$\mathcal{D}\partial_i R(1) \;=\; \mathcal{H}_i\left[F^{(d)} - \frac{r_1^2}{2c^2(d-2)}\partial_t^2 F^{(d)}\right] - \frac{1}{4c^2}\langle n_1^i \mathop{k}_{1}{}_{-3}\rangle + \mathcal{O}\left(\frac{1}{c^4}\right) . \tag{4.30b}$$

These formulae involve some "effective" functions which are to be inserted into the functional brackets of $\mathcal{H}$ and $\mathcal{H}_i$. Beware of the fact that the effective functions are not the same in the cases of a potential and the gradient of that potential. Note the presence, besides the main terms $\mathcal{H}[\cdots]$ and $\mathcal{H}_i[\cdots]$, of some extra terms, purely of order 1PN, in Eqs. (4.30). These terms are made of the average of some coefficients ${}_1k_p$ of the powers r_1^p in the expansion when $r_1 \to 0$ of the *second-time-derivative* of F, namely $\partial_t^2 F$. They do not seem to admit a simple interpretation. They are important to get the final correct result.

C. Distributional derivatives in d dimensions

Let us end this Section by explaining in more details how we dealt with distributional derivatives in d dimensions. First, it is clear that if we were dealing with d-dimensional integrals of the type

$$I \equiv \int d^d\mathbf{x}\,\varphi_{ij}(\mathbf{x})\,\partial_{ij}u_1\,, \tag{4.31}$$

where $\varphi_{ij}(\mathbf{x})$ is some (formally) everywhere smooth function of $\mathbf{x}\in\mathbb{R}^d$, with fast enough decay at infinity, and where $u_1 \equiv \Delta^{-1}\big(-4\pi\delta_1^{(d)}\big)$ is the elementary Newtonian potential in d dimensions [see Eqs. (4.5a) above], we should, in a straightforward d-continuation of Schwartz distributional derivatives, consider that $\partial_{ij}u_1$ contains, besides an "ordinary" singular function $\partial_{ij}(u_1)_{|\mathrm{ord}}$ (treated as a pseudo-function in the sense of Schwartz), a distributional part proportional to $\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)$. In other words, we would write

$$\partial_{ij}(u_1) \;=\; \partial_{ij}(u_1)\Big|_{\mathrm{ord}} - \frac{4\pi}{d}\delta_{ij}\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)\,, \tag{4.32a}$$

$$\partial_{ijkl}(v_1) \;=\; \partial_{ijkl}(v_1)\Big|_{\mathrm{ord}} - \frac{4\pi}{d(d+2)}\Big(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\Big)\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)\,, \tag{4.32b}$$

where the indication "ord" refers to the "ordinary" (pseudo-function) part of the repeated derivative. We have also added the corresponding result for the fourth derivatives of the "less singular" kernel $v_1 \equiv \Delta^{-1}\,u_1$, Eqs. (4.5b). Note that the decompositions above of $\partial_{ij}\,u_1$ or $\partial_{ijkl}\,v_1$ into "ordinary" and "distributional" pieces arise because of our working in (d-dimensional) $\mathbf{x}$-space, and of explicitly computing some derivatives, say as $\partial_{ij}\,(r_1^n)_{|\mathrm{ord}} = [n\,\delta_{ij}+n(n-2)\,n_1^i\,n_1^j]\,r_1^{n-2}$. If we were working in the (d-dimensional) Fourier-transform space $\mathbf{k}$ (which is where dimensional continuation is most clearly defined [34]), the corresponding decomposition would be simply algebraic: *e.g.* $k^ik^j/\mathbf{k}^2 \equiv k^{\langle ij\rangle}/\mathbf{k}^2 + d^{-1}\,\delta_{ij}$, where $k^{\langle ij\rangle}$ denotes the STF part of $k^{ij}\equiv k^ik^j$.

The decompositions (4.32) are clearly needed when dealing with simple integrals of the type (4.31) (with a smooth $\varphi(\mathbf{x})$) to ensure consistency with the requirement that one may integrate by parts (which is one of the defining properties of dim. reg. [34]), and we shall therefore employ it, when applicable. On the other hand, most of the singular integrals that we have to deal with look like (4.31) but contain a *singular* function $\varphi(\mathbf{x})$, of the type of Eq. (4.8). It is, however, a very simplifying feature of dim. reg. that when considering integrals like (4.31) with some *singular* $\varphi(\mathbf{x})$ we can simply ignore any distributional contributions $\propto\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)$ or its derivatives. Indeed, as long as the integer q in the powers $r_1^{p+q\varepsilon}$ present in (4.8) is different from zero (which is precisely the case of all delicate terms involving several propagators shrinking towards a particle world-line), the "singular" expansion (4.8) can be considered, in dim. reg., as defining a sufficiently smooth function [by taking both $q\varepsilon$ and N large enough in (4.8)] which *vanishes*, as well as its derivatives, at $\mathbf{x}=\mathbf{y}_1$. Therefore, all the "dangerous" terms of the form $\mathrm{Sing}_F^{(d)}(\mathbf{x})\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)$ unambiguously vanish in dim. reg.

Let us now consider the consequences of this fact for the time derivatives occurring in expansions such as Eqs. (4.30). The distributional time-derivatives, acting in our present example on u_1 or v_1, *i.e.*, on functions of $r_1^i \equiv x^i - y_1^i(t)$, can be treated in a simple way from the rule $\partial_t = -v_1^i\partial_i$ applicable to the purely distributional part of the derivative. For

instance we can write

$$\partial_t^2(u_1) \;=\; \partial_t^2(u_1)\Big|_{\rm ord} - \frac{4\pi}{d}\mathbf{v}_1^2\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)\,, \tag{4.33a}$$

$$\partial_t^2\partial_{ij}(v_1) \;=\; \partial_t^2\partial_{ij}(v_1)\Big|_{\rm ord} - \frac{4\pi}{d(d+2)}\Big(\delta_{ij}\mathbf{v}_1^2 + 2v_1^iv_1^j\Big)\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)\,. \tag{4.33b}$$

We have checked using these formulae that all the d-dimensional terms coming from second-order derivatives of potentials, taken in the distributional sense (for instance the term $\hat{W}_{ij}\,\partial_{ij}V$ in the source of the $\hat{X}$-potential [22]) yield the *same* purely distributional contributions, in the limit $\varepsilon\to 0$, as the ones that would be computed using what we called above a "pure Schwartz", three-dimensional computation of such contributions [to "smooth" integrals (4.31)]. On the other hand, the extended version of distributional derivatives introduced in [23] does yield some specific additional contributions, two of which were already mentioned in [22] and are reported in the *items* (i) and (ii) of Section III D above, and a third one (also included in [22]) which comes in connection with the second time-derivatives in our formulae for the difference, Eqs. (4.30).

Let us indicate here that the distributional second-time-derivatives in d dimensions have been obtained by using the following (generalizations of) Gel'fand-Shilov formulae [44], valid for general functions $F^{(d)}(\mathbf{x})$ admitting some expansions of the type (4.8): namely, for the spatial derivative,

$$\partial_iF^{(d)} = \partial_iF^{(d)}\Big|_{\rm ord} + \Omega_{d-1}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\big\langle n_1^{iL}\underset{1}{f}{}^{(\varepsilon)}_{-\ell-2,-1}\big\rangle\partial_L\delta_1^{(d)} + 1\leftrightarrow 2\,, \tag{4.34}$$

where $\partial_L\delta_1^{(d)}$ is the ℓ-th partial derivative of the d-dimensional Dirac delta-function at the point 1 ($L\equiv i_1i_2\cdots i_\ell$) and where the angular average is performed over the $(d-1)$-dimensional sphere having total volume Ω_{d-1}; and, concerning the time derivative,

$$\partial_tF^{(d)} = \partial_tF^{(d)}\Big|_{\rm ord} - \Omega_{d-1}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\big\langle n_1^{L}(n_1v_1)\underset{1}{f}{}^{(\varepsilon)}_{-\ell-2,-1}\big\rangle\partial_L\delta_1^{(d)} + 1\leftrightarrow 2\,. \tag{4.35}$$

From the latter formula one deduces the second time-derivative in a way similar to Eqs. (3.43). We have indicated in the *item* (vii) of Section III D the correction it leads to when comparing with the extended-Hadamard prescription for the second time-derivative, and we have subtracted it from $\mathbf{a}_1^{\rm BF}$ to define the pure Hadamard-Schwartz result (3.55). Therefore, we consistently do not need to include such an effect into the differences $\mathcal{D}P(1)$ discussed here.

Finally we are now in position to obtain the supplement of acceleration $\mathcal{D}\mathbf{a}_1$ induced by dimensional regularization, which is composed of the sum of all the differences of potentials and their gradients computed by means of the generals formulae of (4.25), (4.27) and (4.30). The term $\mathcal{D}\mathbf{a}_1$ when added to the "pure-Hadamard-Schwartz" acceleration defined by (3.55), gives our result for the dimensionally regularized ("dr") acceleration

$$\mathbf{a}_1^{\rm dr} = \mathbf{a}_1^{\rm pHS} + \mathcal{D}\mathbf{a}_1 \quad\text{and}\quad 1\leftrightarrow 2\,. \tag{4.36}$$

More details on the practical computation of $\mathcal{D}\mathbf{a}_a$ (which parts of the potentials contribute; what is the diagrammatic picture) will be given in Section V C.

[22] Since this term is to be computed at the 1PN order, not only does it contain second-order derivatives of u_1, but also fourth-order derivatives acting on v_1.

V. DIMENSIONAL REGULARIZATION OF THE EQUATIONS OF MOTION

A. Structure of the dimensionally regularized equations of motion

The preceding Section has explained the method we used to compute the dimensionally regularized equations of motion as the sum ($a = 1, 2$; considered modulo 2)

$$\mathbf{a}_a^{\mathrm{dr}}[\varepsilon, \ell_0] = \mathbf{a}_a^{\mathrm{pHS}}[r'_a, s_{a+1}] + \mathcal{D}\mathbf{a}_a[r'_a, s_{a+1}; \varepsilon, \ell_0]\,, \tag{5.1}$$

where the label "pHS" refers to the "pure Hadamard-Schwartz" definition of the acceleration (*i.e.*, the "raw" result of [22], after subtraction of the additional contributions quoted in Section III D above, Eq. (3.55)), and where $\mathcal{D}\mathbf{a}_a$ is the difference induced when using dimensional continuation as regularization method, instead of Hadamard's one. A first check on our results will be that, as indicated in (5.1), the four regularization parameters (with dimension of length), r'_1, r'_2, s_1, s_2, that enter the Hadamard method must cancel between $\mathbf{a}_a^{\mathrm{pHS}}$ and $\mathcal{D}\mathbf{a}_a$ to leave a result for the dimensionally regularized accelerations $\mathbf{a}_a^{\mathrm{dr}}$ which depends only on the two regularization parameters of dimensional continuation: $\varepsilon \equiv d-3$ and the basic length scale ℓ_0 entering Newton's constant in d dimensions, $G = G_N \ell_0^\varepsilon$, where we recall that G_N denotes the usual three-dimensional Newton constant.

The dimensionally regularized acceleration (5.1) has the structure

$$\begin{aligned}\mathbf{a}_a^{\mathrm{dr}}[\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2] \;&=\; \mathbf{a}_{\mathrm{N}a}[\mathbf{y}_{12}] + \mathbf{a}_{1\mathrm{PN}a}[\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2] \\ &+\; \mathbf{a}_{2\mathrm{PN}a}[\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2] + \mathbf{a}_{2.5\mathrm{PN}a}[\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2] + \mathbf{a}_{3\mathrm{PN}a}[\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2]\,,\end{aligned} \tag{5.2}$$

where we denote $\mathbf{y}_{12} \equiv \mathbf{y}_1 - \mathbf{y}_2$. The 3PN term (which is the only one to have a pole at $\varepsilon = 0$) has a tensor structure of the form (say for the first particle, $a = 1$)

$$A\,\mathbf{n}_{12} + B'\,\mathbf{v}_1 - B''\,\mathbf{v}_2\,, \tag{5.3}$$

where, as usual, $\mathbf{n}_{12} \equiv \mathbf{y}_{12}/r_{12}$ denotes the unit vector directed from particle 2 to particle 1. The scalar coefficients A, B', B'' entering the equation of motion of $\mathbf{y}_1$ can be decomposed in powers of the masses, say

$$A \;=\; \sum_{1\leq n_1+n_2\leq 4} c_{n_1 n_2}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12}, \ln r_{12})\, \frac{G^{n_1+n_2}\, m_1^{n_1}\, m_2^{n_2}}{c^6\, r_{12}^{n_1+n_2+1}}\,, \tag{5.4a}$$

$$B' \;=\; \sum_{1\leq n_1+n_2\leq 3} c'_{n_1 n_2}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12}, \ln r_{12})\, \frac{G^{n_1+n_2}\, m_1^{n_1}\, m_2^{n_2}}{c^6\, r_{12}^{n_1+n_2+1}}\,, \tag{5.4b}$$

$$B'' \;=\; \sum_{1\leq n_1+n_2\leq 3} c''_{n_1 n_2}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12}, \ln r_{12})\, \frac{G^{n_1+n_2}\, m_1^{n_1}\, m_2^{n_2}}{c^6\, r_{12}^{n_1+n_2+1}}\,, \tag{5.4c}$$

where n_1 and n_2 are natural integers, with the restrictions indicated. Note that, in Eqs. (5.4), we have conventionally factored out an integer power of the "full" (d-dimensional) gravitational constant G, and a corresponding integer power of r_{12}. This creates a mismatch between the usual 3-dimensional dimension of, say, $c_{n_1 n_2}^{(d=3)}$ and the dimension of $c_{n_1 n_2}$. Using $G = G_N \ell_0^\varepsilon$ one sees that it is the combination $\ell_0^{(n_1+n_2)\varepsilon} c_{n_1 n_2}$ which has the same dimension as $c_{n_1 n_2}^{(d=3)}$. Alternatively said, the ensuing fact that $r_{12}^{(n_1+n_2)\varepsilon} c_{n_1 n_2}$ has the same dimension as $c_{n_1 n_2}^{(d=3)}$ implies, as indicated in Eqs. (5.4), a dependence of $c_{n_1 n_2}$ on $\ln r_{12}$ when $\varepsilon \to 0$. Notice

also that in Eq. (5.3) we have introduced separate notations for the coefficient of $\mathbf{v}_1$ and that of $\mathbf{v}_2$. Actually, the Poincaré invariance of the equations of motion imposes the restriction $B' = B''$ so that the last two terms in Eq. (5.3) are proportional to the relative velocity $\mathbf{v}_{12} \equiv \mathbf{v}_1 - \mathbf{v}_2$. [Note, however, that B' is not a function of $\mathbf{v}_{12}$ only; it depends both on $\mathbf{v}_1$ and $\mathbf{v}_2$.] Because the calculation of the separate contributions $\mathbf{a}_a^{\rm pHS}$ and $\mathcal{D}\mathbf{a}_a$ to the equations of notion breaks the over-all Poincaré invariance of the formalism, our computation of the separate pieces $\mathbf{a}_a^{\rm pHS}$ and $\mathcal{D}\mathbf{a}_a$ will involve partial contributions to B' and B'' that do not coincide. It is only at the end of the calculation that the equality $B \equiv B' = B''$ will be satisfied, so that finally

$$\mathbf{a}_{\rm 3PN1} = A\,\mathbf{n}_{12} + B\,\mathbf{v}_{12}\,. \tag{5.5}$$

Most of the coefficients $c_{n_1n_2}$, $c'_{n_1n_2}$, $c''_{n_1n_2}$ entering the 3PN acceleration are well behaved when $\varepsilon \to 0$, in the sense that their evaluation never involves any poles $\propto 1/\varepsilon$. By this we mean that whatever be the (reasonable) way of decomposing the integral giving a coefficient in separate contributions, the latter contributions do not involve poles $\propto 1/\varepsilon$. The subset of coefficients whose evaluation involves poles coincides with the set of "delicate" coefficients in the Hadamard regularization, namely the nine coefficients contributing to terms of the following form in the acceleration of the first particle:

$$\begin{aligned}
&\frac{G^4}{c^6\,r_{12}^5}\Big[c_{31}\,m_1^3\,m_2 + c_{22}\,m_1^2\,m_2^2 + c_{13}\,m_1\,m_2^3\Big]\mathbf{n}_{12}\\
&\quad+\frac{G^3\,m_1^2\,m_2}{c^6\,r_{12}^4}\Big[c_{21}(\mathbf{v}_1,\mathbf{v}_2)\mathbf{n}_{12} + c'_{21}(\mathbf{v}_1,\mathbf{v}_2)\mathbf{v}_1 - c''_{21}(\mathbf{v}_1,\mathbf{v}_2)\,\mathbf{v}_2\Big]\\
&\quad+\frac{G^3\,m_2^3}{c^6\,r_{12}^4}\Big[c_{03}(\mathbf{v}_1,\mathbf{v}_2)\,\mathbf{n}_{12} + c'_{03}(\mathbf{v}_1,\mathbf{v}_2)\,\mathbf{v}_1 - c''_{03}(\mathbf{v}_1,\mathbf{v}_2)\,\mathbf{v}_2\Big]\,.
\end{aligned} \tag{5.6}$$

The first three terms in Eq. (5.6) do not depend on velocities and will be referred to as the *static* delicate contributions, by contrast to the *kinetic* delicate contributions involving the velocity-dependent coefficients c_{21}, c'_{21}, c''_{21}, c_{03}, c'_{03}, and c''_{03} (they depend on $\mathbf{v}_1$, $\mathbf{v}_2$ and also on $\mathbf{n}_{12}$).

B. Pure-Hadamard-Schwartz static contributions to the equations of motion

Correspondingly to the decomposition (5.1) of the equations of motion, the dimensionally regularized static contributions[23] $c_{31}^{\rm dr}$, $c_{22}^{\rm dr}$, $c_{13}^{\rm dr}$ to the acceleration of the first particle can be written as the sum ($m+n=4$, $m\geq 1$, $n\geq 1$)

$$c_{mn}^{\rm dr}[\varepsilon] = c_{mn}^{\rm pHS}[r'_1, s_2] + \mathcal{D}c_{mn}[r'_1, s_2, \varepsilon]\,. \tag{5.7}$$

In this subsection, we discuss the explicit evaluation of the pure Hadamard-Schwartz static coefficients $c_{mn}^{\rm pHS}$.

As explained in the previous Section, the pHS static contributions $c_{mn}^{\rm pHS}[r'_1, s_2]$ are obtained from the results reported in [22] by undoing two things. First, the "BF" results reported there for $\mathbf{a}_1$ (Eq. (7.16) of [22]) were expressed in terms of the three parameters r'_1, r'_2 and

[23] As explained above, we consider only the "delicate" ones. In the present case, this means that we do not consider the contribution $c_{04}^{\rm dr} = 16 + \mathcal{O}(\varepsilon)$, unambiguously obtained from the test-mass limit $m_1 \ll m_2$.

λ, instead of the two pure Hadamard parameters r'_1 and s_2 more relevant for the present purpose. The introduction of the parameter λ was motivated by requiring that the full set of equations of motion (which *a priori* depended on four independent regularizing parameters r'_1, r'_2, s_1, s_2) admit a conserved energy. This led to the link, Eqs. (7.9) in [22], [24]

$$\ln\left(\frac{r'_2}{s_2}\right) = \frac{159}{308} + \lambda\,\frac{m_1+m_2}{m_2}\,. \tag{5.8}$$

When inserting (5.8) in the expression of $\mathbf{a}_1^{\rm BF}[r'_1, r'_2, \lambda]$ we find, as it should be, that the result simplifies to an expression depending only on the two pure Hadamard parameters r'_1 and s_2. This leads to the following net results from [22],

$$c_{31}^{\rm BF}[r'_1, s_2] \;=\; -\frac{3187}{1260} + \frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right), \tag{5.9a}$$

$$c_{22}^{\rm BF}[r'_1, s_2] \;=\; \frac{34763}{210} - \frac{41}{16}\,\pi^2\,, \tag{5.9b}$$

$$c_{13}^{\rm BF}[r'_1, s_2] \;=\; \frac{1565}{9} - \frac{41}{16}\,\pi^2 - \frac{44}{3}\ln\left(\frac{r_{12}}{s_2}\right). \tag{5.9c}$$

Second, Ref. [22] obtained their results for the equations of motion by adding to the pure Hadamard-Schwartz contributions 7 additional corrections, imposed by their extended-Hadamard regularization and explained in Section III D above: see the *items* (i)-(vii) there. Note that these various corrections affect the "delicate" contributions to $\mathbf{a}_1$, in general both the static and kinetic ones, but only five of them contribute to the static part. These are the self term (3.48), the Leibniz term (3.49), the V-correction given by (3.50), the EOM non-distributivity (3.52), and the distributional time-derivative one (3.54). Following Eq. (3.55), and focussing on the static contributions, we now *subtract* these static terms from the result (5.9) in order to get the looked-for pure Hadamard contributions:

$$c_{31}^{\rm pHS}[r'_1, s_2] \;=\; c_{31}^{\rm BF}[r'_1, s_2] - \frac{779}{210}\,, \tag{5.10a}$$

$$c_{22}^{\rm pHS}[r'_1, s_2] \;=\; c_{22}^{\rm BF}[r'_1, s_2] + \frac{97}{210}\,, \tag{5.10b}$$

$$c_{13}^{\rm pHS}[r'_1, s_2] \;=\; c_{13}^{\rm BF}[r'_1, s_2] - 5 + \frac{88}{9} - \frac{151}{9} + \frac{2}{15}\,, \tag{5.10c}$$

i.e., explicitly,

$$c_{31}^{\rm pHS}[r'_1, s_2] \;=\; -\frac{1123}{180} + \frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right), \tag{5.11a}$$

$$c_{22}^{\rm pHS}[r'_1, s_2] \;=\; 166 - \frac{41}{16}\,\pi^2\,, \tag{5.11b}$$

$$c_{13}^{\rm pHS}[r'_1, s_2] \;=\; \frac{7291}{45} - \frac{41}{16}\,\pi^2 - \frac{44}{3}\ln\left(\frac{r_{12}}{s_2}\right). \tag{5.11c}$$

[24] We use here the link corresponding to the "particular" improved distributional derivative $\mathrm{D}_i[F]$ given by (3.38) above. Another derivative, the "correct" one, was also considered in [22] and shown to yield equivalent equations of motion. The pure Hadamard result does not depend on this choice because we shall subtract below the specific additional contributions coming from the distributional derivative $\mathrm{D}_i[F]$.

Note in passing that though the coefficient c_{22} does not contain regularization logarithms, its evaluation involves many intermediate logarithmic divergencies that cancel in the final result. Such "cancelled logs" lead to as much ambiguity in the final result than uncancelled ones that explicitly depend on an arbitrary regularization scale such as r'_1 or s_2 in c_{31} or c_{13}.

C. Dimensionally regularized static contributions

We now turn to the evaluation of the "dim.-reg. minus pure-Hadamard" differences $\mathcal{D}c_{mn}$ in Eq. (5.7), coming from the differences $\mathcal{D}\mathbf{a}_1$ in Eq. (5.1). We start from the d-dimensional expression for the acceleration $\mathbf{a}_1$ [see Eq. (2.23) for a short-hand form], which is itself expressed in terms of the d-dimensional elementary potentials V, V_i, K, $\hat{W}_{ij}$, $\hat{R}_i$, $\hat{X}$, $\hat{Z}_{ij}$, $\hat{Y}_i$ and $\hat{T}$ defined in Section II. Each elementary potential can be naturally decomposed in a "compact" (or, equivalently, "contact") piece (whose source is compact, *i.e.*, involves the basic delta-function sources σ, σ_i, σ_{ij}) and a "non-compact" one (whose source is non-linearly generated and extends all over space). The potentials V, V_i and K are purely "compact", $V = V^C$, $V_i = V_i^C$, $K = K^C$, while all the other potentials admit a decomposition of the form $\hat{W}_{ij} = \hat{W}_{ij}^C + W_{ij}^{NC}$, etc. For instance, the "compact" part of $\hat{W}_{ij}$ is defined by

$$\Box\, \hat{W}_{ij}^C = -4\pi G \left(\sigma_{ij} - \frac{1}{d-2}\, \delta_{ij}\, \sigma_{kk} \right), \tag{5.12}$$

while its "non-compact" part is defined by

$$\Box\, \hat{W}_{ij}^{NC} = -\frac{1}{2}\,\frac{d-1}{d-2}\, \partial_i V\, \partial_j V\,. \tag{5.13}$$

A more complicated example is the potential $\hat{X} = \hat{X}^C + \hat{X}^{NC}$ with

$$\Box\, \hat{X}^C = -4\pi G \left[\frac{1}{d-2}\, V \sigma_{ii} + 2\,\frac{d-3}{d-1}\, \sigma_i\, V_i + \left(\frac{d-3}{d-1}\right)^2 \sigma \left(\frac{1}{2} V^2 + K \right) \right], \tag{5.14}$$

and

$$\begin{aligned} \Box\, \hat{X}^{NC} \;&=\; \hat{W}_{ij}\, \partial_{ij} V - 2\, \partial_i\, V_j\, \partial_j\, V_i + 2\, V_i\, \partial_t\, \partial_i\, V \\ &+\; \frac{1}{2}\,\frac{d-1}{d-2}\, V \partial_t^2\, V + \frac{d(d-1)}{4(d-2)^2}\, (\partial_t\, V)^2\,. \end{aligned} \tag{5.15}$$

The NC contribution can be further decomposed into the piece of $\hat{X}^{NC}$ whose source is quadratic in compact potentials, namely

$$\Box\, \hat{X}^{VV} = \hat{W}_{ij}^C\, \partial_{ij}\, V - 2\, \partial_i\, V_j\, \partial_j\, V_i + \text{ other } VV \text{ terms}\,, \tag{5.16}$$

and its "cubically non-compact" piece given by

$$\Box\, \hat{X}^{CNC} = \hat{W}_{ij}^{NC}\; \partial_{ij}\, V = \Box^{-1} \left(-\frac{1}{2}\,\frac{d-1}{d-2}\, \partial_i\, V \partial_j\, V \right) \partial_{ij}\, V\,. \tag{5.17}$$

To get a feeling of the actual evaluation of the difference $\mathcal{D}\mathbf{a}_1$ let us consider a specific contribution to $\mathbf{a}_1$, say the term

$$a_1^i[\hat{X}] \equiv \frac{4}{c^4}\,(\partial_i\,\hat{X})_{\mathbf{x}=\mathbf{y}_1}\,. \tag{5.18}$$

It can be decomposed into: (i) its "compact" piece $\mathbf{a}_1[\hat{X}^C]$, (ii) its "quadratically non-compact" one $\mathbf{a}_1[\hat{X}^{VV}]$, (iii) its "cubically non-compact" part $\mathbf{a}_1[\hat{X}^{CNC}]$.

It is sometimes convenient to think of the various contributions to $\mathbf{a}_1$ in terms of spacetime diagrams. If we represent the basic delta-function sources [proportional to $m_1\,\delta(\mathbf{x}-\mathbf{y}_1)$ and $m_2\,\delta(\mathbf{x}-\mathbf{y}_2)$] as two world-lines and each propagator $\Box^{-1}$ as a dotted line, a "compact" contribution to $\mathbf{a}_1$ will be represented by one of the diagrams in Fig. 1. For instance, Fig. 1a

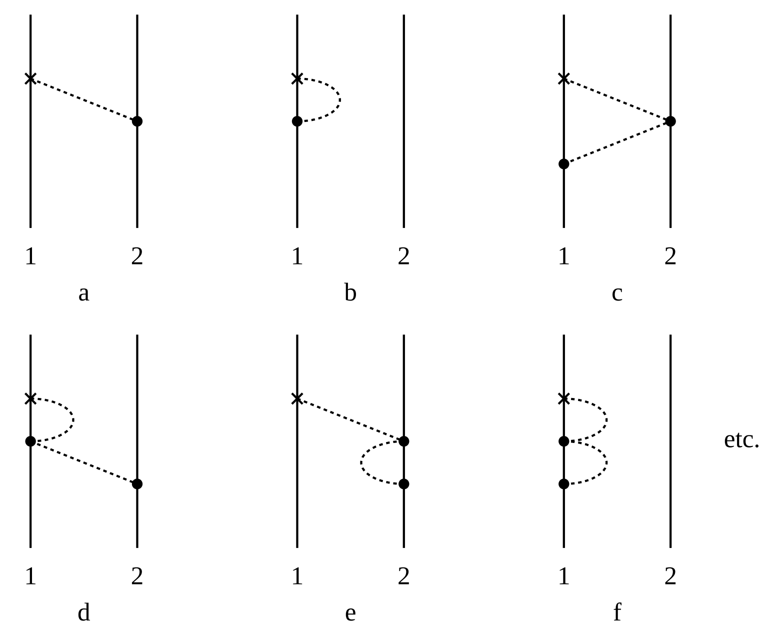

FIG. 1: Diagrams representing "compact" contributions to acceleration $\mathbf{a}_1$. The dotted line represents $\Box^{-1}$, the cross represents the field point $\mathbf{x}$ (here taken on the first worldline), and the bullet represents either a source point or (in the Figures below) an intermediate nonlinear vertex.

can represent a term $(\partial_i\,V)_1$ in a_1^i in which the (compact) source σ of V is proportional to $m_2\,\delta(\mathbf{x}-\mathbf{y}_2)$ and involves no further powers of the masses, while Fig. 1b represents a *self-action* term[25] in $(\partial_i\,V)_1$ with source proportional to $m_1\,\delta(\mathbf{x}-\mathbf{y}_1)$. By contrast, Fig. 1c might correspond to another term in $(\partial_i\,V)_1$ where the compact source σ is concentrated at

[25] While in usual regularization schemes using dimensionful cut-offs (*e.g.* small length scales s_1, s_2) the self-action diagrams, such as Fig. 1b or Fig. 1d, are the first divergencies that one encounters and must then renormalize away, dimensional regularization has the technically useful property of setting all of these diagrams to zero. Indeed, when using a time-symmetric propagator $\Box^{-1}_{\rm sym} = \Delta^{-1} + c^{-2}\,\partial_t^2\,\Delta^{-2} + \cdots$ these diagrams are seen to involve the coinciding-point limits of $|\mathbf{x}-\mathbf{y}_1|^{2-d+2n}$, which vanish when $\mathbf{x}\to\mathbf{y}_1$ by dimensional continuation in d.

$\mathbf{y}_2$, $\sigma_2 = \tilde{\mu}_2\, \delta(\mathbf{x}-\mathbf{y}_2)$, and where a part of the "effective mass",

$$\tilde{\mu}_2 = \frac{2}{d-1}\, \frac{m_2\, c}{\sqrt{-(g_{\rho\sigma})_2\, v_2^\rho\, v_2^\sigma}}\, \frac{(d-2)+\mathbf{v}_2^2/c^2}{\sqrt{-(g)_2}}\,, \tag{5.19}$$

contains, besides the overall factor m_2, another factor m_1. As all the sources of $\hat{X}^C$ contain, besides some "basic" $\sigma_{\mu\nu}$, a potential (V, V_i, V^2 or K), the diagrams contained in $a_1^i[\hat{X}^C]$ will be at least of the form of Fig. 1c, 1d, 1e, 1f, or will involve a more complicated mass-dependence.

The quadratically non-compact terms $a_1^i[\hat{X}^{VV}]$ will then contain diagrams of the type of Fig. 2, while the cubically non-compact term $a_1^i[\hat{X}^{CNC}]$ contains many subdiagrams of the type sketched in Fig. 3.

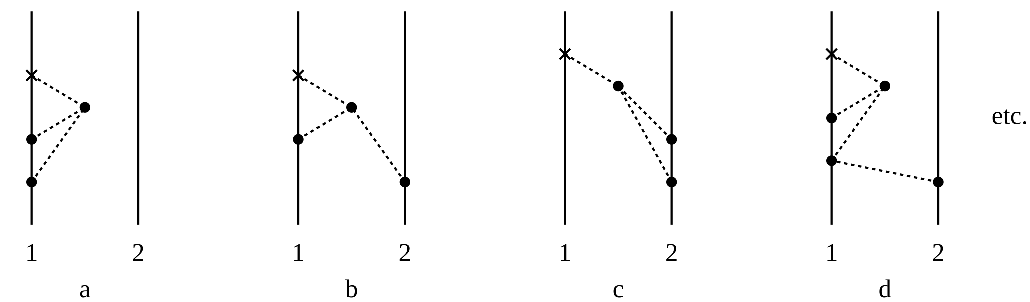

FIG. 2: Quadratically non-compact contributions to acceleration $\mathbf{a}_1$.

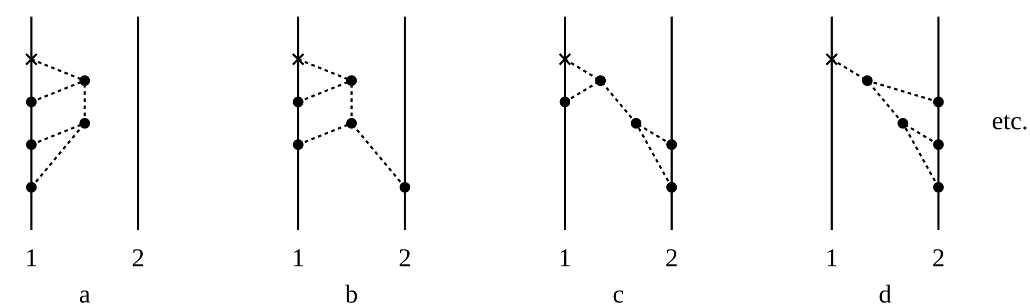

FIG. 3: Cubically non-compact contributions to acceleration $\mathbf{a}_1$.

The particular term (5.18) that we considered contains only diagrams of the general type of Figs. 1, 2 or 3. Note, however, that there are also more non-linear contributions to a_1^i, such as some terms in

$$a_1^i[\hat{T}] = \frac{16}{c^6}\, (\partial_i \hat{T})_1\,, \tag{5.20}$$

corresponding to diagrams of the type sketched in Fig. 4. Similarly to the diagrams Fig. 1c or Fig. 2d, all the diagrams above can be modified by the presence of additional lines propagating directly between the two world-lines and corresponding to "potential" modifications of compact-support sources.

As underlined in Section IV A above, the 3PN equations of motion do *not* involve "quartically non-linear" contributions corresponding to diagrams such as those of Fig. 5. Terms

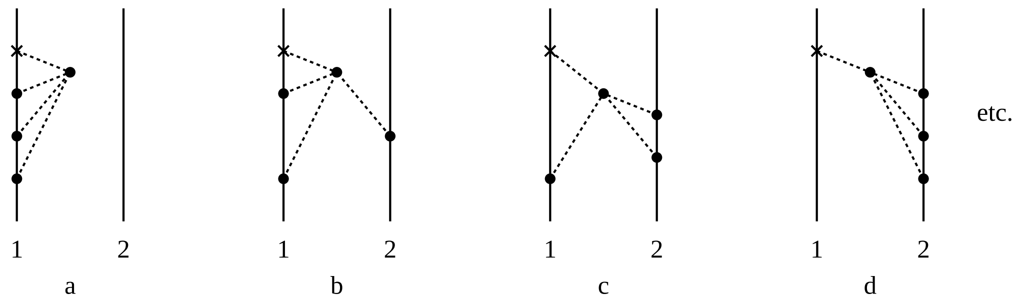

FIG. 4: Other non-linear contributions to acceleration $\mathbf{a}_1$.

like $\Delta^{-1}\left(V^2\partial_k V\partial_k V\right)$ or $\Delta^{-1}\left(\partial_k V\partial_k \hat{X}\right)$ are of this form, and they do occur in the 3PN acceleration $\mathbf{a}_1$, but since they involve double contracted gradients, it was possible to integrate them away thanks to rule (ii) of Section II; see Eq. (A12) in Appendix A below. On the other hand, terms of the form $\partial_i V\partial_j V\hat{W}_{ij}$ or $\partial_i \hat{W}_{jk}\partial_j \hat{W}_{ki}$ do not occur at the 3PN order $\mathcal{O}(1/c^6)$, although they are of the third post-*Minkowskian* order $\mathcal{O}(G^4)$.

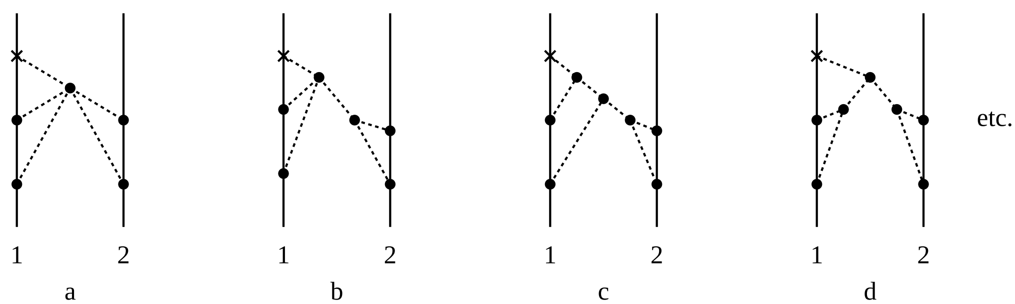

FIG. 5: Quartically non-compact contributions which do *not* occur in our calculation of acceleration $\mathbf{a}_1$ at the 3PN order.

Drawing diagrams often helps to highlight the nature of the UV singularities contained in the integrals they represent. As a rule of thumb, the "delicate" diagrams, that might involve poles, or cancelled poles, when $\varepsilon \to 0$ (corresponding to logarithms, or cancelled logarithms, in $d = 3$) are characterized by the presence of a subdiagram containing three propagator lines that can simultaneously shrink to zero size, as a subset of vertices coalesce together on one of the two world-lines. Examples of such UV dangerous diagrams are Fig. 2d and Fig. 3b [for vertices coalescing towards $\left(t, \mathbf{y}_1(t)\right)$] or Fig. 3d and Fig. 4d [for vertices coalescing on the second world-line]. The former diagrams can give poles proportional to $m_1^2 m_2$ (with some velocity dependence, or some extra mass dependence due to an extra line propagating between the two world-lines), while the latter can give poles proportional to m_2^3 (possibly with some extra velocity or mass dependence). The reason why three simultaneously shrinking propagators can yield poles as $\varepsilon \to 0$ is easy to see in the approximation where the relativistic propagators $\Box^{-1}$ are replaced by non-relativistic ones $\Delta^{-1}_{\mathbf{x},\mathbf{x}'} = -\frac{\tilde{k}}{4\pi}|\mathbf{x}-\mathbf{x}'|^{-1-\varepsilon}$. Indeed, when three such propagators shrink simultaneously, the overall integral contains a subintegral of the form $\int d^{3+\varepsilon}\mathbf{x}\,(|\mathbf{x}|^{-1-\varepsilon})^3 \sim \int_0^a \mathrm{d}r\, r^{-1-2\varepsilon} \sim a^{-2\varepsilon}/(-2\varepsilon)$.

On the other hand, beyond our obtaining a heuristic feeling of what are the origins of the poles in $\mathbf{a}_1$, we did not use a diagrammatic technique for evaluating the equations of

TABLE I: Static contributions of $\mathcal{D}\partial_i\hat{X}(1)$. All the results are presented modulo some neglected terms $\mathcal{O}(\varepsilon)$. The "principal" part of a term corresponds to the term Δ^{-1} in the 1PN symmetric propagator $\Box^{-1}_{\rm 1PN} = \Delta^{-1} + c^{-2}\partial_t^2\Delta^{-2}$, while the "retarded" part corresponds to the purely 1PN piece $c^{-2}\partial_t^2\Delta^{-2}$. The "extra term" refers to the last term in the R.H.S. of Eq. (4.30b). Note that, in view of Eqs. (5.21), one must multiply the results by a factor 4 in order to get the contributions to the coefficients $\mathcal{D}c_{mn}$ in the equations of motion.

	$\frac{1}{4}\,\mathcal{D}c_{31}$	$\frac{1}{4}\,\mathcal{D}c_{22}$	$\frac{1}{4}\,\mathcal{D}c_{13}$
$\hat{W}_{ij}\,\partial_{ij}V\vert_{\rm principal}$	$\frac{4}{5}+\frac{1}{6\varepsilon}-\frac{1}{3}\ln_r$	$\frac{103}{200}+\frac{1}{20\varepsilon}-\frac{1}{10}\ln_r$	$-\frac{5}{18}-\frac{1}{12\varepsilon}+\frac{1}{6}\ln_s$
$\hat{W}_{ij}\,\partial_{ij}V\vert_{\rm retarded}$	$-\frac{25}{18}-\frac{1}{3\varepsilon}+\frac{2}{3}\ln_r$	$-\frac{2947}{1800}-\frac{23}{60\varepsilon}+\frac{23}{30}\ln_r$	$-\frac{53}{90}-\frac{1}{12\varepsilon}+\frac{1}{6}\ln_s$
$\frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2V\vert_{\rm principal}$	$\frac{11}{18}+\frac{1}{6\varepsilon}-\frac{1}{3}\ln_r$	$\frac{11}{18}+\frac{1}{6\varepsilon}-\frac{1}{3}\ln_r$	0
$\frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2V\vert_{\rm retarded}$	$\frac{11}{18}+\frac{1}{6\varepsilon}-\frac{1}{3}\ln_r$	$\frac{11}{18}+\frac{1}{6\varepsilon}-\frac{1}{3}\ln_r$	0
extra term	$-\frac{1}{6}$	$-\frac{13}{60}$	0
Total	$\frac{7}{15}+\frac{1}{6\varepsilon}-\frac{1}{3}\ln_r$	$-\frac{7}{60}$	$-\frac{13}{15}-\frac{1}{6\varepsilon}+\frac{1}{3}\ln_s$

motion. [Note, however, that a generalization of the (2PN level) work [45] would lead to a diagrammatic technique for evaluating the Fokker Lagrangian of two point masses.] Our actual computations used the techniques elaborated in the previous Sections.

We evaluated the contributions to the difference $\mathcal{D}\mathbf{a}_1$ coming from all the terms in the expression for $\mathbf{a}_1$ deduced from (2.19) together with the complete expanded forms (A11)-(A12). However, as expected from various arguments — diagrammatic analysis, existence of (possibly cancelled) logarithms in the corresponding $d=3$ evaluation — most of the terms lead to a vanishing difference $\mathcal{D}\mathbf{a}_1$. The only terms that give non-vanishing contributions to $\mathcal{D}\mathbf{a}_1$ are the four terms given in Eq. (2.23),

$$\begin{aligned} a_1^i[\hat{X}] &= \frac{4}{c^4}\,(\partial_i\,\hat{X})_1\,, \quad a_1^i[\hat{T}] = \frac{16}{c^6}\,(\partial_i\,\hat{T})_1\,, \\ a_1^i[\hat{R}_i] &= \frac{8}{c^4}\,\frac{d}{dt}\,(\hat{R}_i)_1\,, \quad a_1^i[\hat{Y}_i] = \frac{16}{c^6}\,\frac{d}{dt}\,(\hat{Y}_i)_1\,. \end{aligned} \tag{5.21}$$

Note that, for the contributions associated to $\hat{X}$ and $\hat{R}_i$, one needs a 1PN-accurate treatment of both their respective sources and the propagator $\Box^{-1}$. Apart from the compact support terms in the sources for $\hat{X}$, $\hat{T}$, $\hat{R}_i$ and $\hat{Y}_i$ which lead to zero difference, most of the non-compact terms do lead to some non-vanishing contributions to the difference of acceleration $\mathcal{D}\mathbf{a}_1$. We give in Tables I-IV the contributions to $\mathcal{D}c_{mn}$ associated to the various individual source terms of the "delicate" potentials $\hat{X}$, $\hat{T}$, $\hat{R}_i$ and $\hat{Y}_i$, which were displayed in Section II, Eqs. (2.12) [of course, we limit ourselves to non-compact source terms]. In these tables, we use the simplifying notation

$$\ln_r \equiv \ln\left(\overline{q}\,r'_1 r_{12}\right)\,, \qquad \ln_s \equiv \ln\left(\overline{q}\,s_2 r_{12}\right)\,, \qquad \overline{q} \equiv 4\pi e^C\,, \tag{5.22}$$

where $C = 0.577\ldots$ denotes the Euler constant.

Summing up the separate non-vanishing contributions displayed in Tables I-IV, we get

TABLE II: Static contributions of $\mathcal{D}\partial_i \hat{T}(1)$.

	$\frac{1}{16}\,\mathcal{D}c_{31}$	$\frac{1}{16}\,\mathcal{D}c_{22}$	$\frac{1}{16}\,\mathcal{D}c_{13}$
$\hat{Z}_{ij}\partial_{ij}V$	$-\frac{119}{900}-\frac{1}{30\varepsilon}+\frac{1}{15}\ln_r$	$\frac{1429}{900}+\frac{11}{30\varepsilon}-\frac{11}{15}\ln_r$	$\frac{28}{9}+\frac{2}{3\varepsilon}-\frac{4}{3}\ln_s$
$\frac{1}{8}\left(\frac{d-1}{d-2}\right)^2 V^2\partial_t^2 V$	$-\frac{19}{36}-\frac{1}{6\varepsilon}+\frac{1}{3}\ln_r$	$-\frac{119}{450}-\frac{1}{15\varepsilon}+\frac{2}{15}\ln_r$	$-\frac{19}{36}-\frac{1}{6\varepsilon}+\frac{1}{3}\ln_s$
$-\frac{1}{2}\left(\partial_t V_i\right)^2$	0	0	0
$-\frac{(d-1)(d-3)}{4(d-2)^2}\,V\partial_t^2 K$	0	0	0
$-\frac{(d-1)(d-3)}{4(d-2)^2}\,K\partial_t^2 V$	0	0	0
$-\frac{1}{2}\left(\frac{d-3}{d-2}\right)\hat{W}_{ij}\partial_{ij}K$	0	0	0
Total	$-\frac{33}{50}-\frac{1}{5\varepsilon}+\frac{2}{5}\ln_r$	$\frac{397}{300}+\frac{3}{10\varepsilon}-\frac{3}{5}\ln_r$	$\frac{31}{12}+\frac{1}{2\varepsilon}-\ln_s$

TABLE III: Static contributions of $\mathcal{D}\frac{d\hat{R}_i}{dt}(1)$. The "principal" part, "retarded" part and the "extra term" have the same meaning as in Table I.

	$\frac{1}{8}\,\mathcal{D}c_{31}$	$\frac{1}{8}\,\mathcal{D}c_{22}$	$\frac{1}{8}\,\mathcal{D}c_{13}$
$-\frac{d-1}{d-2}\,\partial_k V\partial_i V_k\vert_{\text{principal}}$	$-\frac{31}{18}-\frac{1}{3\varepsilon}+\frac{2}{3}\ln_r$	$-\frac{31}{18}-\frac{1}{3\varepsilon}+\frac{2}{3}\ln_r$	0
$-\frac{d-1}{d-2}\,\partial_k V\partial_i V_k\vert_{\text{retarded}}$	$\frac{43}{18}+\frac{1}{3\varepsilon}-\frac{2}{3}\ln_r$	$\frac{43}{18}+\frac{1}{3\varepsilon}-\frac{2}{3}\ln_r$	0
$-\frac{d(d-1)}{4(d-2)^2}\,\partial_t V\partial_i V\vert_{\text{principal}}$	$\frac{5}{4}+\frac{1}{4\varepsilon}-\frac{1}{2}\ln_r$	$\frac{5}{4}+\frac{1}{4\varepsilon}-\frac{1}{2}\ln_r$	0
$-\frac{d(d-1)}{4(d-2)^2}\,\partial_t V\partial_i V\vert_{\text{retarded}}$	$-\frac{7}{4}-\frac{1}{4\varepsilon}+\frac{1}{2}\ln_r$	$-\frac{7}{4}-\frac{1}{4\varepsilon}+\frac{1}{2}\ln_r$	0
extra term	$-\frac{1}{4}$	$-\frac{1}{4}$	0
Total	$-\frac{1}{12}$	$-\frac{1}{12}$	0

the following total differences

$$\mathcal{D}c_{31} = -\frac{22}{3\varepsilon}+\frac{44}{3}\ln(\overline{q}\,r'_1\,r_{12})-\frac{102}{5}+\mathcal{O}(\varepsilon)\,, \tag{5.23a}$$

$$\mathcal{D}c_{22} = 9+\mathcal{O}(\varepsilon) \tag{5.23b}$$

$$\mathcal{D}c_{13} = \frac{22}{3\varepsilon}-\frac{44}{3}\ln(\overline{q}\,s_2\,r_{12})+\frac{568}{15}+\mathcal{O}(\varepsilon)\,. \tag{5.23c}$$

Finally, adding Eqs. (5.23) to the pure Hadamard-Schwartz result (5.11), we get the dimensionally regularized static contributions to $\mathbf{a}_1$:

$$c_{31}^{\text{dr}} = -\frac{22}{3\varepsilon}+\frac{44}{3}\ln(\overline{q}\,r_{12}^2)-\frac{959}{36}+\mathcal{O}(\varepsilon)\,, \tag{5.24a}$$

$$c_{22}^{\text{dr}} = 175-\frac{41}{16}\pi^2+\mathcal{O}(\varepsilon)\,, \tag{5.24b}$$

$$c_{13}^{\text{dr}} = \frac{22}{3\varepsilon}-\frac{44}{3}\ln(\overline{q}\,r_{12}^2)+\frac{1799}{9}-\frac{41}{16}\pi^2+\mathcal{O}(\varepsilon)\,. \tag{5.24c}$$

TABLE IV: Static contributions of $\mathcal{D}\frac{d\hat{Y}_i}{dt}(1)$.

	$\frac{1}{16}\,\mathcal{D}c_{31}$	$\frac{1}{16}\,\mathcal{D}c_{22}$	$\frac{1}{16}\,\mathcal{D}c_{13}$
$\hat{W}_{kl}\,\partial_{kl}V_i$	0	0	0
$-\frac{1}{2}\left(\frac{d-1}{d-2}\right)\partial_t\hat{W}_{ik}\,\partial_k V$	$\frac{65}{18}+\frac{2}{3\varepsilon}-\frac{4}{3}\ln_r$	$\frac{107}{45}+\frac{5}{12\varepsilon}-\frac{5}{6}\ln_r$	$\frac{71}{450}+\frac{1}{60\varepsilon}-\frac{1}{30}\ln_s$
$\partial_i\hat{W}_{kl}\,\partial_k V_l$	$\frac{257}{450}+\frac{7}{60\varepsilon}-\frac{7}{30}\ln_r$	$-\frac{149}{225}-\frac{2}{15\varepsilon}+\frac{4}{15}\ln_r$	$\frac{71}{450}+\frac{1}{60\varepsilon}-\frac{1}{30}\ln_s$
$-\partial_k\hat{W}_{il}\,\partial_l V_k$	$-\frac{257}{450}-\frac{7}{60\varepsilon}+\frac{7}{30}\ln_r$	$\frac{149}{225}+\frac{2}{15\varepsilon}-\frac{4}{15}\ln_r$	$-\frac{71}{450}-\frac{1}{60\varepsilon}+\frac{1}{30}\ln_s$
$-\frac{d-1}{d-2}\,\partial_k V\,\partial_i\hat{R}_k$	$\frac{2681}{900}+\frac{19}{30\varepsilon}-\frac{19}{15}\ln_r$	$\frac{3791}{900}+\frac{53}{60\varepsilon}-\frac{53}{30}\ln_r$	$-\frac{71}{450}-\frac{1}{60\varepsilon}+\frac{1}{30}\ln_s$
$-\frac{d(d-1)}{4(d-2)^2}\,V_k\,\partial_i V\partial_k V$	$-\frac{53}{100}-\frac{1}{10\varepsilon}+\frac{1}{5}\ln_r$	$-\frac{9}{2}-\frac{1}{\varepsilon}+2\ln_r$	$-\frac{33}{100}-\frac{1}{10\varepsilon}+\frac{1}{5}\ln_s$
$-\frac{d(d-1)^2}{8(d-2)^3}\,V\partial_t V\partial_i V$	$-\frac{9}{4}-\frac{1}{2\varepsilon}+\ln_r$	$\frac{43}{25}+\frac{2}{5\varepsilon}-\frac{4}{5}\ln_r$	$\frac{33}{100}+\frac{1}{10\varepsilon}-\frac{1}{5}\ln_s$
$-\frac{1}{2}\left(\frac{d-1}{d-2}\right)^2 V\partial_k V\partial_k V_i$	0	0	0
$\frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2 V_i$	$-\frac{9}{2}-\frac{1}{\varepsilon}+2\ln_r$	$-\frac{9}{2}-\frac{1}{\varepsilon}+2\ln_r$	0
$2V_k\,\partial_k\partial_t V_i$	0	0	0
$\frac{(d-1)(d-3)}{(d-2)^2}\,\partial_k K\partial_i V_k$	0	0	0
$\frac{d(d-1)(d-3)}{4(d-2)^3}\,\partial_t V\partial_i K$	0	0	0
$\frac{d(d-1)(d-3)}{4(d-2)^3}\,\partial_i V\partial_t K$	0	0	0
Total	$-\frac{69}{100}-\frac{3}{10\varepsilon}+\frac{3}{5}\ln_r$	$-\frac{69}{100}-\frac{3}{10\varepsilon}+\frac{3}{5}\ln_r$	0

As expected the two Hadamard regularization length scales r'_1 and s_2 have cancelled between c^{pHS}_{mn} and $\mathcal{D}c_{mn}$ to leave a result which depends only on the dim. reg. regularization parameter $\varepsilon = d-3$. One might be surprised by the presence in c^{dr}_{mn} of terms $\pm\frac{44}{3}\ln(r_{12}^2)$ compared to corresponding terms $\pm\frac{44}{3}\ln(r_{12})$ in c^{pHS}_{mn}, and by the absence of any adimensionalizing length scale in these logarithms of r_{12}^2. These two properties can be understood when one remembers from the discussion above that the coefficients which have the same physical dimension as $c^{(d=3)}_{mn}$ are the combinations $\ell_0^{(m+n)\varepsilon}\,c^{\mathrm{dr}}_{mn}$. In the present case, this means that $\ell_0^{4\varepsilon}\,c^{\mathrm{dr}}_{mn}$ are dimensionless. It is easy to see, thanks to the pole terms $\mp 22/(3\varepsilon)$ and the expansion $\ell_0^{4\varepsilon}\equiv\exp(4\varepsilon\ln\ell_0) = 1+2\varepsilon\ln\ell_0^2+\mathcal{O}(\varepsilon^2)$, that the combinations $\ell_0^{4\varepsilon}\,c^{\mathrm{dr}}_{31}$ and $\ell_0^{4\varepsilon}\,c^{\mathrm{dr}}_{13}$ do indeed depend only on the dimensionless quantities ε and $\ln(r_{12}^2/\ell_0^2)$.

VI. RENORMALIZATION OF THE EQUATIONS OF MOTION

A. Poles in the dimensionally regularized bulk metric

The first computation of the dimensional continuation of the 3PN gravitational interaction of point masses was done in ADM coordinates and resulted into a *finite* (*i.e.*, without $1/\varepsilon$ poles) answer [35]. Our task in analyzing the physical meaning of the harmonic-coordinates result (5.24) is to interpret the presence of $1/\varepsilon$ poles in it. For this we have to remem-

ber that, as in Quantum Field Theory (QFT), dimensional continuation is a *regularization* method which, like all regularization methods, transforms truly infinite results, say containing $\int_0^{r_{12}} d^3\mathbf{x}/r_1^3$, into finite, but "large" ones, which depend on some cut-off parameter, *e.g.* $\int_{s_1}^{r_{12}} d^3\mathbf{x}/r_1^3 = 4\pi \ln(r_{12}/s_1)$ or $\int_0^{r_{12}} d^{3+\varepsilon}\mathbf{x}/r_1^{3(1+\varepsilon)} \propto 1/\varepsilon$. Any regularization must be followed by a *renormalization* process which allows one to absorb the cut-off dependent terms in some of the basic *bare parameters* of the theory.

In order to have a clearer understanding of the poles in the (static) equations of motion (5.24) [we shall prove below that our discussion extends to the full, velocity-dependent equations of motion], we need to analyze the presence of poles in the "bulk" metric, *i.e.*, the metric $g_{\mu\nu}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)$ evaluated at a generic field point $\mathbf{x}$, away from the two world-lines. Indeed, if we were considering the gravitational field generated by regular (*i.e.*, non point-like) sources, a complete physical description of their gravitational effects would necessitate the simultaneous consideration of the bulk metric and of the equations of motion of the (extended) sources. Similarly, in the present formal study of two-point-like sources, we need to consider both the equations of motion $\ddot{\mathbf{y}}_a = \mathbf{a}_a(\mathbf{y}_1, \mathbf{y}_2, \mathbf{v}_1, \mathbf{v}_2)$ and the bulk metric $g_{\mu\nu}\big(\mathbf{x}; \mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{v}_1(t), \mathbf{v}_2(t)\big)$.

It is here that the diagrammatic representation introduced above plays a useful role in highlighting the structure of divergencies in the equations of motion and in the bulk metric. Indeed, it is clear that the divergent diagrams of the equations of motion of the first particle, where the $1/\varepsilon$ pole is due to the presence of a subdivergence induced by three propagators shrinking onto the second world-line (such as in Fig. 3d or Fig. 4d), will correspond to similar $1/\varepsilon$ poles in the bulk metric, for the corresponding "bulk diagrams" where the special point marked by a cross in the diagrams above (denoting the coincidence $\mathbf{x} = \mathbf{y}_1$) is detached from the first world-line to end at an arbitrary point in the bulk, as indicated in Fig. 6.

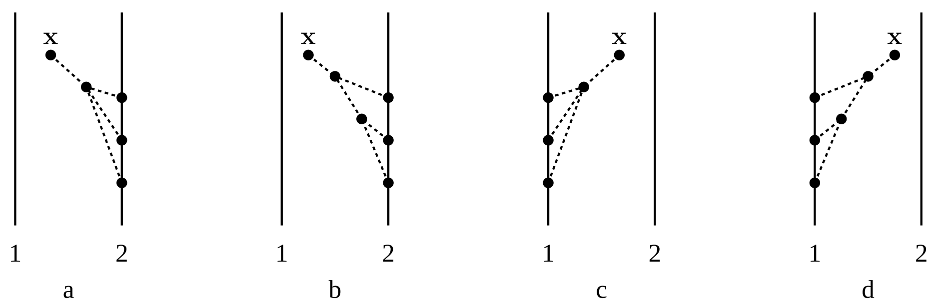

FIG. 6: Some divergent diagrams for the bulk metric. Here, contrary to the previous Figures, the field point, labelled by an $\mathbf{x}$, is detached from the first worldline.

Evidently, in addition to such diagrams as Fig. 6a and 6b which will contain (at least) a factor m_2^3, there will exist "mirror diagrams", containing a factor m_1^3, and obtained by exchanging the labels 1 and 2. On the other hand, note that the bulk poles $\propto m_1^3$ of the type of Fig. 6c and 6d do not (necessarily) correspond to poles in the equations of motion of $\mathbf{y}_1$ because their coincidence limits $\mathbf{x} \to \mathbf{y}_1$ induce diagrams of the type of Fig. 3a or 4a containing four shrinking propagators instead of three. [Though such diagrams would exhibit worse divergences in a dimensionful cut-off regularization schemes, they are generally non dangerous in dimensional regularization because the integral $\int d^{3+\varepsilon}\mathbf{x}/r_1^{4(1+\varepsilon)}$ has no pole as $\varepsilon \to 0$.]

A careful analysis of the possible presence of poles in the various potentials $V, V_i, \cdots, \hat{T}$

we use to parametrize the bulk metric (aided by the structure of potentially dangerous terms sketched in Fig. 6) shows that, at the 3PN approximation,[26] such poles can only be present in the 1PN-level expansion of $\hat{X}$ and in the Newtonian-level approximation of $\hat{T}$. Drawing on the results of [22] and [37] we can also see that all velocity-dependent terms in the poles present in $\hat{X}_{\rm 1PN}$ and $\hat{T}$ (*i.e.*, the terms proportional to $m_1^3\, v_1^2$ or $m_2^3\, v_2^2$) exactly cancel in the combination $4\,\hat{X}/c^4 + 16\,\hat{T}/c^6$ that matters for the bulk metric. [This shows up, for instance, in Eq. (7.1) of [22] which implies that the divergencies linked to the second world-line, characterized by the presence of $\ln s_2$, do not depend on velocities. This shows up also in the absence of poles in the velocity-dependent contributions to $\mathbf{a}_1$ proportional to m_2^3, see Eqs. (6.47c)-(6.47d) below.] We are therefore left with evaluating the poles present in the *static limit* ($\mathbf{v}_1, \mathbf{v}_2 \to 0$) of $\hat{X}_{\rm 1PN}$ and $\hat{T}$. Clearly, from Fig. 6, the poles in $\hat{X}_{\rm 1PN}$ and $\hat{T}$ will come only from cubically non-compact (CNC) sources. Finally, as we are only interested in the pole part we can neglect the d-dependence of the coefficients in the sources of $\hat{X}$ and $\hat{T}$ (which we indicate by using a symbol $\simeq$). Thus, these poles can only come from

$$\begin{aligned}
\hat{X}^{CNC}_{\rm static} &= \Box^{-1}\Big[\partial_{ij}\, V\, \hat{W}^{NC}_{ij}\Big] \\
&\simeq \Box^{-1}\Big[\partial_{ij}\, V\, \Box^{-1}\big(-\partial_i\, V \partial_j\, V\big)\Big]\,, \qquad (6.1a) \\
\hat{T}^{CNC}_{\rm static} &\simeq \Box^{-1}\left[\frac{1}{2}\, V^2\, \partial_t^2\, V + \partial_{ij}\, V\, \hat{Z}^{NC}_{ij}\right] \\
&\simeq \Box^{-1}\left[\frac{1}{2}\, V^2\, \partial_t^2\, V + \partial_{ij}\, V\, \Box^{-1}\big(-2\partial_i\, V \partial_t\, V_j\big)\right]. \qquad (6.1b)
\end{aligned}$$

The static poles (involving factors m_1^3 or m_2^3) in Eqs. (6.1) are then obtained by: (1) considering sources involving three times V_1 or three times V_2 (where V_a denotes the piece $\propto m_a$ in V), (2) evaluating the time derivatives in the static limit, using for instance

$$(\partial_t^2\, V_a)_{\rm static} = -a_a^j\, \partial_j\, V_a\,, \qquad (6.2)$$

and (3) expanding up to the required accuracy the (time-symmetric) propagators according to $\Box^{-1} = \Delta^{-1} + c^{-2}\, \partial_t^2\, \Delta^{-2} + \mathcal{O}(c^{-4})$.

As an example among the simplest terms, let us consider the m_1^3 contribution coming from the first term on the R.H.S. of Eq. (6.1b),

$$\hat{T}^{m_1^3}_{\rm static(1)} \simeq \frac{1}{2}\, \Delta^{-1}[V_1^2\, \partial_t^2\, V_1]_{\rm static} = -\frac{1}{2}\, \Delta^{-1}[V_1^2\, a_1^j\, \partial_j\, V_1] = -\frac{1}{6}\, a_1^j\, \partial_j\, \Delta^{-1}[V_1^3]\,. \qquad (6.3)$$

Using $V_1 = 2\frac{d-2}{d-1}\, G\, \tilde{k}\, m_1\, r_1^{2-d} + \mathcal{O}(c^{-2}) \simeq G\, m_1\, r_1^{-1-\varepsilon} + \mathcal{O}(c^{-2})$ and $\Delta^{-1}\, r_1^\lambda = r_1^{\lambda+2}/[(\lambda+2)(\lambda+d)]$, one finds that the pole part of (6.3) reads

$$\hat{T}^{m_1^3}_{\rm static(1)} \simeq -\frac{1}{12\varepsilon}\, G^3\, m_1^3\, a_1^j\, \partial_j\, r_1^{-1-3\varepsilon}\,. \qquad (6.4)$$

Similarly an analysis of the second source term in Eq. (6.1b) yields

$$\hat{T}^{m_1^3}_{\rm static(2)} \simeq \frac{1}{3\varepsilon}\, G^3\, m_1^3\, a_1^j\, \partial_j\, r_1^{-1-3\varepsilon}\,, \qquad (6.5)$$

[26] *I.e.*, at order c^{-8} in g_{00}, c^{-7} in g_{0i}, and c^{-6} in g_{ij}.

so that the full (static) contribution of $\hat{T}$ is

$$\hat{T}^{m_1^3}_{\rm static} \simeq \frac{1}{4\varepsilon}\, G^3\, m_1^3\, a_1^j\, \partial_j\, r_1^{-1-3\varepsilon}\,. \tag{6.6}$$

The analysis of the pole part in the static limit of $\hat{X}$, Eq. (6.1a), is more intricate because one must expand to 1PN accuracy both $V_a \simeq G\, m_a\, r_a^{-1-\varepsilon} + \frac{1}{2c^2}\, G\, m_a\, \partial_t^2\, r_a^{1-\varepsilon}$ and the propagator $\Box^{-1}$. This yields

$$\hat{X}^{m_1^3}_{\rm static} \simeq -\frac{1}{12\varepsilon}\, \frac{G^3}{c^2}\, m_1^3\, a_1^j\, \partial_j\, r_1^{-1-3\varepsilon}\,. \tag{6.7}$$

Let us now consider the improved V-potential (2.10) that makes up the essential part of g_{00},

$$\mathcal{V} \equiv V - \frac{2}{c^2}\left(\frac{d-3}{d-2}\right) K + \frac{4\hat{X}}{c^4} + \frac{16\hat{T}}{c^6} \simeq V + \frac{4\hat{X}}{c^4} + \frac{16\hat{T}}{c^6}\,, \tag{6.8}$$

such that $g_{00} = -\exp(-2\mathcal{V}/c^2)[1 - 8\, V_i\, V_i/c^6 - 32\, R_i\, V_i/c^8 + \mathcal{O}(1/c^{10})]$, see Eq. (2.11a). Combining the results above, we find that the only $1/\varepsilon$ poles in the bulk metric $g_{\mu\nu}(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$ show up in g_{00} at the 3PN level and are (when expressed in terms of the improved potential $\mathcal{V}$, and after cancellation of $m_a^3\, v_a^2/\varepsilon$ terms between $\hat{X}$ and $\hat{T}$ of the following static form

$$\mathcal{V}(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2) = V + \frac{1}{c^2}\mathcal{V}_2 + \frac{1}{c^4}\mathcal{V}_4 + \frac{1}{c^6}\left[\mathcal{V}_6' + \frac{11}{3\varepsilon}\sum_a G^3\, m_a^3\, a_a^j\, \partial_j\, r_a^{-1-3\varepsilon}\right], \tag{6.9}$$

where $\mathcal{V}_2$, $\mathcal{V}_4$,$\mathcal{V}_6'$'s are finite when $\varepsilon \to 0$. To understand better the structure of result (6.9) let us introduce the notation

$$\zeta_a^i \equiv +\frac{11}{3\varepsilon}\, \frac{G_N^2\, m_a^2}{c^6}\, a_a^i\,, \tag{6.10}$$

where G_N is the 3-dimensional Newton constant and a_a^i the d-dimensional acceleration of y_a^i. [This definition ensures that ζ_a^i has the physical dimension of a length.] In terms of the definition (6.10), the result (6.9) can be equivalently written as

$$\mathcal{V}(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2) = \sum_a \Big[V_a(\mathbf{x} - \mathbf{y}_a) + \zeta_a^j\, \partial_j\, V_a(\mathbf{x} - \mathbf{y}_a)\Big] + \frac{1}{c^2}\mathcal{V}_2 + \frac{1}{c^4}\mathcal{V}_4 + \frac{1}{c^6}\mathcal{V}_6\,, \tag{6.11}$$

where the pole part is entirely contained in the terms proportional to ζ_1^j and ζ_2^j [$\mathcal{V}_6$ here differs from $\mathcal{V}_6'$ in Eq. (6.9) by some finite corrections when $\varepsilon \to 0$]. The fact that poles appear only in $\mathcal{V}$, at order c^{-6}, implies that there are no divergencies in the harmonic gauge conditions (2.13) in the bulk. Indeed, (2.13a) needs $\mathcal{V}$ at order c^{-4} only, and (2.13b) at Newtonian order only.

B. Renormalization of poles by shifts of the world-lines

Result (6.11) indicates a simple way of renormalizing away the poles present in the bulk metric. Indeed, the logic up to now has been to describe in the simplest possible manner a gravitationally interacting two-particle system, parametrized by the following *bare* parameters: $G^{\rm bare}$, $m_1^{\rm bare}$, $m_2^{\rm bare}$, $\mathbf{y}_1^{\rm bare}$, $\mathbf{y}_2^{\rm bare}$, considered in everywhere-harmonic coordinates, $\Gamma^\lambda \equiv g^{\alpha\beta}\, \Gamma^\lambda_{\alpha\beta} = 0$. In particular, the internal structure of each particle has been, up to now,

entirely described by a monopolar stress-energy distribution, *i.e.*, $T_a^{\mu\nu} \propto m_a^{\rm bare}\, \delta(\mathbf{x}-\mathbf{y}_a^{\rm bare})$. In other words, we have set to zero any higher multipolar structure. Eq. (6.11) is most simply interpreted by saying that the non-linear interactions (see Fig. 6) dress each particle by a cloud of gravitational energy which generates, at the 3PN order, a divergent *dipole* in the Newtonian-like potential. Therefore, to get a net, finite bulk gravitational field we must endow each initial particle by an infinite, bare dipole, corresponding to a counterterm $\Delta T_a^{\mu\nu} \propto -m_a^{\rm bare} \zeta_a^j\, \partial_j\, \delta(\mathbf{x}-\mathbf{y}_a)$, which will cancel the non-linearly generated one (6.11). An equivalent, but technically simpler way of endowing each particle by a bare structure able to cancel the dipolar pole terms in (6.11) is simply to say that the central *bare* world-lines used in our derivations up to now, henceforth denoted as $\mathbf{y}_a^{\rm bare}$, can be decomposed in a finite *renormalized* part $\mathbf{y}_a^{\rm ren}$ and a formally infinite shift $\boldsymbol{\xi}_a$ involving a pole $\propto 1/\varepsilon$,

$$\mathbf{y}_a^{\rm bare} \equiv \mathbf{y}_a^{\rm ren} + \boldsymbol{\xi}_a\,, \quad \mathbf{v}_a^{\rm bare} \equiv \mathbf{v}_a^{\rm ren} + \dot{\boldsymbol{\xi}}_a\,. \tag{6.12}$$

The gravitational potential of two point particles $[\propto \delta(\mathbf{x}-\mathbf{y}_a^{\rm bare})]$ is then

$$\begin{aligned}\mathcal{V}(\mathbf{x},\mathbf{y}_1^{\rm bare},\mathbf{y}_2^{\rm bare}) \;=\; & \sum_a \Big[V_a(\mathbf{x}-\mathbf{y}_a^{\rm ren}-\boldsymbol{\xi}_a) + \zeta_a^j\,\partial_j\, V_a(\mathbf{x}-\mathbf{y}_a^{\rm ren}-\boldsymbol{\xi}_a)\Big] \\ & + \frac{1}{c^2}\,\mathcal{V}_2(\mathbf{x},\mathbf{y}_a^{\rm ren}+\boldsymbol{\xi}_a) + \frac{1}{c^4}\,\mathcal{V}_4(\mathbf{x},\mathbf{y}_a^{\rm ren}+\boldsymbol{\xi}_a) + \frac{1}{c^6}\,\mathcal{V}_6(\mathbf{x},\mathbf{y}_a^{\rm ren}+\boldsymbol{\xi}_a)\,.\end{aligned} \tag{6.13}$$

Assuming that the vector $\boldsymbol{\xi}_a$ is of 3PN order [*i.e.*, $\boldsymbol{\xi}_a = \mathcal{O}(1/c^6)$], we can rewrite Eq. (6.13) as

$$\begin{aligned}\mathcal{V}(\mathbf{x},\mathbf{y}_1^{\rm bare},\mathbf{y}_2^{\rm bare}) \;=\; & \sum_a \Big[V_a(\mathbf{x}-\mathbf{y}_a^{\rm ren}) + (\zeta_a^j-\xi_a^j)\,\partial_j\, V_a(\mathbf{x}-\mathbf{y}_a^{\rm ren})\Big] \\ & + \frac{1}{c^2}\,\mathcal{V}_2(\mathbf{x},\mathbf{y}_a^{\rm ren}) + \frac{1}{c^4}\,\mathcal{V}_4(\mathbf{x},\mathbf{y}_a^{\rm ren}) + \frac{1}{c^6}\,\mathcal{V}_6(\mathbf{x},\mathbf{y}_a^{\rm ren}) + \mathcal{O}\left(\frac{1}{c^8}\right),\end{aligned} \tag{6.14}$$

which makes it clear that the potential will be finite (at 3PN accuracy) when $\varepsilon \to 0$ if we choose

$$\boldsymbol{\xi}_a = \boldsymbol{\zeta}_a + \mathcal{O}\left(\frac{\varepsilon^0}{c^6}\right), \tag{6.15}$$

where by $\mathcal{O}(\varepsilon^0/c^6)$ we mean a term finite when $\varepsilon \to 0$ and of the 3PN order. We shall henceforth refer to $\boldsymbol{\xi}_a$ in Eq. (6.15) as a *shift* of the $a^{\rm th}$ world-line. The reasoning above shows that the introduction of such shifts, at the 3PN order and having the pole structure (6.10), is *necessary* to renormalize away the poles present in the *bulk metric*. It remains to show that these shifts are also *sufficient* to renormalize away the poles present in the *equations of motion*.

The effect of 3PN-level shifts $\boldsymbol{\xi}_a$ on the equations of motion is easy to obtain. Indeed, the equations of motion we computed above concern the original, *bare* world-lines $\mathbf{y}_a^{\rm bare}$. For the first particle, they had the structure (in dimensional regularization)

$$\begin{aligned}\ddot{\mathbf{y}}_1^{\rm bare} \;&=\; \mathbf{a}_1^{\rm dr}(\mathbf{y}_{12}^{\rm bare},\mathbf{v}_1^{\rm bare},\mathbf{v}_2^{\rm bare}) \\ &=\; \mathbf{a}_{\rm N1}^{\rm dr}(\mathbf{y}_{12}^{\rm bare}) + \mathbf{a}_{\rm 1PN1}^{\rm dr}(\mathbf{y}_{12}^{\rm bare},\mathbf{v}_1^{\rm bare},\mathbf{v}_2^{\rm bare}) \\ &\;+\; \mathbf{a}_{\rm 2PN1}^{\rm dr}(\mathbf{y}_{12}^{\rm bare},\mathbf{v}_1^{\rm bare},\mathbf{v}_2^{\rm bare}) + \mathbf{a}_{\rm 2.5PN1}^{\rm dr}(\mathbf{y}_{12}^{\rm bare},\mathbf{v}_1^{\rm bare},\mathbf{v}_2^{\rm bare}) \\ &\;+\; \mathbf{a}_{\rm 3PN1}^{\rm dr}(\mathbf{y}_{12}^{\rm bare},\mathbf{v}_1^{\rm bare},\mathbf{v}_2^{\rm bare})\,,\end{aligned} \tag{6.16}$$

where $\mathbf{y}_{12}^{\rm bare} \equiv \mathbf{y}_1^{\rm bare} - \mathbf{y}_2^{\rm bare}$. Here $\mathbf{a}_{\rm N1}^{\rm dr}$ denotes the dimensionally continued Newtonian-level acceleration,

$$a_{\rm N1}^{i({\rm dr})} = \partial_i \, V_2(\mathbf{y}_{12}) = f \, G \, m_2 \, \tilde{k} \, \partial_i \, r_{12}^{2-d} \,, \tag{6.17}$$

where by a slight abuse of notation we pose $\partial_i = \partial / \partial \, y_{12}^i$, where $G \equiv G_N \, \ell_0^\varepsilon$ denotes the d-dimensional gravitational constant, and where the d-dependent correcting factors

$$f \equiv 2 \, \frac{d-2}{d-1} = \frac{1+\varepsilon}{1+\varepsilon/2} \,, \qquad \tilde{k} \equiv \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} = \frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)}{\pi^{\frac{1+\varepsilon}{2}}} \,, \tag{6.18}$$

tend to 1 as $\varepsilon \to 0$, but will play a significant role below.

When inserting the redefinitions (6.12) into (6.16) one easily finds that the *renormalized* equations of motion, *i.e.*, the equations for $\mathbf{y}_a^{\rm ren}$, read [using only $\boldsymbol{\xi}_a = \mathcal{O}(1/c^6)$ at this stage]

$$\ddot{\mathbf{y}}_1^{\rm ren} = \mathbf{a}_1^{\rm ren}(\mathbf{y}_{12}^{\rm ren}, \mathbf{v}_1^{\rm ren}, \mathbf{v}_2^{\rm ren}) \,, \tag{6.19}$$

where

$$\mathbf{a}_1^{\rm ren}(\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2) = \mathbf{a}_1^{\rm dr}(\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2) + \delta_{\boldsymbol{\xi}} \, \mathbf{a}_1(\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2) + \mathcal{O}\left(\frac{1}{c^8}\right) , \tag{6.20}$$

with

$$\delta_{\boldsymbol{\xi}} \, \mathbf{a}_1(\mathbf{y}_{12}, \mathbf{v}_1, \mathbf{v}_2) = (\xi_1^j - \xi_2^j) \, \partial_j \, \mathbf{a}_{\rm N1}^{\rm dr}(\mathbf{y}_{12}) - \ddot{\boldsymbol{\xi}}_1 \,. \tag{6.21}$$

Let us note that the effect on the equations of motion of a (3PN) shift of the world-lines, Eq. (6.21), is technically identical to the effect on the equations of motion of the restriction to the world-lines of a 3PN-level coordinate transformation, say $x_{\rm new}^i = x^i - \epsilon^i(\mathbf{x}, t)$ and $t_{\rm new} = t - c^{-1} \epsilon^0(\mathbf{x}, t)$. Indeed, a coordinate transformation has two effects: (i) it changes the bulk metric by $\delta_\epsilon \, g_{\mu\nu}(x) = \mathcal{L}_\epsilon \, g_{\mu\nu}(x)$, where $\mathcal{L}_\epsilon$ denotes the Lie derivative along ϵ^μ, and (ii) it induces a shift of the world-lines $y_{a\,\rm new}^i = y_a^i - \epsilon_a^i + c^{-1} \epsilon_a^0 v_a^i$ (plus non-linear terms in ϵ^μ), where we denote the coordinate change at the location of the a-th particle by $\epsilon_a^i(t) \equiv [\epsilon^i(\mathbf{x}, t)]_{\mathbf{x}=\mathbf{y}_a}$. Because of the diffeomorphism invariance of the total action, the effect (i) does not change the action,[27] so that the net effect of a coordinate transformation on the equations of motion reduces to the effect (6.21) of the following shift induced on the world-lines:

$$\xi_a^i = \epsilon_a^i - c^{-1} \epsilon_a^0 \, v_a^i \ + \text{non-linear terms} \,. \tag{6.22}$$

The coordinate transformations considered in [22], see Eq. (6.11) there, were of the type $\epsilon_\mu(x) = c^{-6} \, \partial_\mu\big(\sum_a \kappa_a / r_a\big)$, where the κ_a's are some coefficients, so that we see that the latter induced shift reduces at the 3PN order to the (regularization of the) purely spatial coordinate transformation evaluated on the world-line:

$$\xi_a^i = \epsilon_a^i + \mathcal{O}\left(\frac{1}{c^8}\right) . \tag{6.23}$$

We have checked that the formula given by Eq. (6.15) in [22] for the coordinate transformation of the acceleration of the particle 1 gives exactly the same result as the one computed

[27] Actually, one should consider, as *e.g.* in [45], a non-generally-covariant gauge-fixed action. But the "double zero" nature of the gauge-fixing term, say $\propto \sqrt{-g} \, g_{\mu\nu} \, \Gamma^\mu \, \Gamma^\nu$, ensures that it does not contribute to first order in ϵ^μ.

from the effect of the shift (6.21). [The agreement extends to d dimensions if we consider the straightforward extension of the latter coordinate transformation ϵ^μ to d dimensions, namely $\epsilon_\mu(x) = c^{-6}\,\partial_\mu\bigl(\sum_a \kappa_a\,\tilde{k}\,r_a^{2-d}\bigr)$.]

Note that the coordinate transformations $\epsilon^\mu(x)$ were considered in [22] only in terms of their effects, Eqs. (6.22)-(6.23), on the equations of motion. This was sufficient to prove, for instance, that the two constants r'_1 and r'_2 in the 3PN equations of motion are not physical, because they can be gauged away in 3 dimensions and therefore will never appear in any physical result. However, we remark that the extension to d dimensions of the coordinate transformation $\epsilon^\mu(x)$ of the *bulk* metric, say $\epsilon_\mu(x) = c^{-6}\,\partial_\mu\bigl(\sum_a \kappa_a\,\tilde{k}\,r_a^{2-d}\bigr)$ (with coefficients $\kappa_a \propto \varepsilon^{-1}$, as needed to remove the poles in the equations of motion), does not lead to a bulk metric free of poles. Indeed, assuming $\kappa_a \propto \varepsilon^{-1}$, we see that the pole in the spatial coordinate transformation $\epsilon_i(x)$ would then induce a pole in the spatial components of the metric, $\delta_\epsilon\, g_{ij} = \partial_i\epsilon_j + \partial_j\epsilon_i + \cdots$, but this is inadmissible because we have proved above Eqs. (6.1) that, at 3PN order, only the time-time component of the bulk metric contained a pole. A bulk coordinate transformation of the type above can then remove the poles in the time-time component of the bulk metric only at the price of creating a pole in the initially pole-free spatial metric. We shall leave to future work a complete clarification of the possibility of using, within our dim. reg. context, a coordinate transformation to induce the shifts (6.15). For the time being, what is important is that our introduction above of shifts of the world-lines (*a priori* unconnected to any coordinate transformation) is a consistent way of renormalizing away the poles in the metric, and that its effect on the equations of motion, Eq. (6.21), is identical to the transformations of the acceleration obtained in Ref. [22].

It remains now to show that the same world-line shifts (6.15) that renormalize away the poles in the bulk metric, Eq. (6.14), do renormalize away also the poles present in the original bare equations of motion [see (5.24) for the static contributions and (6.47) below for the kinetic ones]. For this purpose let us consider a shift of the more general form

$$\xi_a^i = \frac{e(d)}{\varepsilon}\,\frac{G_N^2\,m_a^2}{c^6}\,a_a^i\,, \tag{6.24}$$

where $e(d)$ represents a certain numerical coefficient depending on d, and where a_a^i denotes the d-dimensional acceleration of y_a^i given by its Newtonian approximation (6.17) [but, for notational simplicity, we henceforth drop the label N on such accelerations entering 3PN effects]. Inserting (6.24) into (6.21) yields (for the index $a=1$)

$$\delta_\xi\, a_1^i = \frac{e(d)}{\varepsilon}\,\frac{G_N^2}{c^6}\Bigl[(m_1^2 - m_2^2)\,a_2^j\,\partial_j\,a_1^i - \,m_1^2\,v_{12}^{jk}\,\partial_{jk}\,a_1^i\Bigr]\,, \tag{6.25}$$

where $v_{12}^j \equiv v_1^j - v_2^j$ and $v_{12}^{jk} \equiv v_{12}^j v_{12}^k$ [and also, as before, $\partial_j \equiv \partial/\partial y_{12}^j$]. Before further evaluating (6.25) by inserting the explicit expression (6.17) for the acceleration, we shall consider some simple but important consequences of the structure (6.25).

C. Link to the general class of harmonic equations of motion

As recalled in the Introduction, previous work on the 3PN equations of motion in *harmonic coordinates* has shown that these equations necessarily belonged to a three-parameter class of equations of motion, say

$$\ddot{\mathbf{y}}_a^{(d=3)} = \mathbf{a}_a^{\rm BF}(\mathbf{y}_{12},\mathbf{v}_1,\mathbf{v}_2;\lambda,r'_1,r'_2)\,. \tag{6.26}$$

The dimensionless parameter λ could not be determined by the previous work in harmonic coordinates. However, comparison with the work in ADM coordinates, has shown [20, 25] that, *if there were consistency* between the two calculations one should have the following link between λ and the corresponding ADM "static ambiguity" parameter ω_s,

$$\lambda = -\frac{3}{11}\,\omega_s - \frac{1987}{3080}\,. \tag{6.27}$$

If dimensional regularization is a fully consistent regularization scheme for classical perturbative gravity, we then expect that the dim. reg. determination of ω_s in ADM coordinates [35], namely $\omega_s^{\text{dr ADM}} = 0$, should lead to a dim. reg. direct determination of λ (in harmonic coordinates) of $\lambda^{\text{dr harmonic}} = -1987/3080$. We will turn to this verification in a moment.

The two other parameters, denoted above r'_1, r'_2, entering the general "parametric" harmonic equations of motion (6.26) have the dimension of length and have the character of gauge parameters. Indeed, they can be chosen at will (except that one cannot set them to zero) by the effect of shifts of the world-line, induced for instance [but not necessarily, *cf.* a discussion in Section VI B above] by some gauge transformations. In the way they were originally introduced [22], the two parameters r'_1 and r'_2 can be interpreted as some infinitesimal radial distances used as cut-offs when the field point tends towards the two singularities $\mathbf{y}_1$ and $\mathbf{y}_2$. Therefore in principle $\ln r'_1$ and $\ln r'_2$ should initially be thought of as being (formally) infinite. However, it is trivial to show that by a (formally infinite) gauge transformation, involving the logarithmic ratios $\ln(r''_1/r'_1)$ and $\ln(r''_2/r'_2)$, where r''_1 and r''_2 denote any two *finite* length scales, one can replace everywhere r'_1, r'_2 by the finite scales r''_1, r''_2. By this process it is therefore possible to identify the two sets of scales and thereby to think of the scales r'_1, r'_2 as being in fact finite, as implicitly done in Ref. [22]. In the language of renormalization theory, the original (infinitesimal) scales r'_1 and r'_2 would be referred to as *Hadamard-regularization* scales entering the computation of divergent Poisson integrals [see Section III B above], while the (finite) scales r''_1 and r''_2 would be referred to as the arbitrary *renormalization* scales entering the final, renormalized harmonic-coordinates equations of motion. In the present paper, in order to remain close to the notation used in [22], we shall keep the notation r'_1 and r'_2, but interpret them as arbitrary finite constants, which means that we shall *identify* them with the finite renormalization length scales r''_1 and r''_2. In other words, the scales r'_1, r'_2 used in the present Section should in principle be distinguished from the scales r'_1, r'_2 used in Section III B above. [Remember, in this respect, that the regularization scales r'_1, r'_2 have disappeared when computing the dim. reg. equations of motion.]

With our notation, and still focussing on the static contributions to the equations of motion, the "parametric" equations of motion (6.26) imply the following structure for the static coefficients c_{mn}:

$$c_{31}^{\text{BF}}(\lambda, r'_1, r'_2) = \frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right) - \frac{3187}{1260}\,, \tag{6.28a}$$

$$c_{22}^{\text{BF}}(\lambda, r'_1, r'_2) = \frac{34763}{210} - \frac{44}{3}\,\lambda - \frac{41}{16}\,\pi^2\,, \tag{6.28b}$$

$$c_{13}^{\text{BF}}(\lambda, r'_1, r'_2) = -\frac{44}{3}\ln\left(\frac{r_{12}}{r'_2}\right) + \frac{10478}{63} - \frac{44}{3}\,\lambda - \frac{41}{16}\,\pi^2\,. \tag{6.28c}$$

It will be convenient to replace the parameter λ by the parameter ω_s, using (6.27) as a defining one-to-one map between λ and ω_s. With this change of notation the static coefficients

become

$$
\begin{aligned}
c_{31}^{\mathrm{BF}}(\omega_s, r'_1, r'_2) &= \frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right) - \frac{3187}{1260}\,, &\text{(6.29a)}\\
c_{22}^{\mathrm{BF}}(\omega_s, r'_1, r'_2) &= 175 + 4\,\omega_s - \frac{41}{16}\,\pi^2\,, &\text{(6.29b)}\\
c_{13}^{\mathrm{BF}}(\omega_s, r'_1, r'_2) &= -\frac{44}{3}\ln\left(\frac{r_{12}}{r'_2}\right) + \frac{110741}{630} + 4\,\omega_s - \frac{41}{16}\,\pi^2\,. &\text{(6.29c)}
\end{aligned}
$$

Note that there are two combinations of the three coefficients c_{mn}^{BF} which do not depend on $\ln r_{12}$, namely c_{22}^{BF}, and the combination $c_{31}^{\mathrm{BF}} + c_{13}^{\mathrm{BF}}$, or even better the combination

$$
c_{31}^{\mathrm{BF}} + c_{13}^{\mathrm{BF}} - c_{22}^{\mathrm{BF}} = \frac{44}{3}\ln\left(\frac{r'_2}{r'_1}\right) - \frac{7}{4}\,, \tag{6.30}
$$

which depends neither on $\ln r_{12}$ nor on ω_s (or λ), and which contains, like for c_{22}^{BF}, simpler looking rational numbers.

We now come back to the effect of the general shift (6.24) on the dim. reg. equations of motion. Let us first focus on the static terms. We recall that the (dim. reg.) *renormalized* equations of motion had necessarily the form (6.20). By projecting the latter equation along the static terms c_{mn}, with $m + n = 4$ [recalling Eq. (5.6)], it will induce a result for the *renormalized* static coefficients of the form

$$
c_{mn}^{\mathrm{ren}} = c_{mn}^{\mathrm{dr}}(\varepsilon) + \delta_{\boldsymbol{\xi}(\varepsilon)} c_{mn}\,, \tag{6.31}
$$

where the $\delta_{\boldsymbol{\xi}(\varepsilon)} c_{mn}$'s are the static coefficients corresponding to $\delta_{\boldsymbol{\xi}}\, a_1^i$, Eq. (6.21). When choosing $\xi_a^i(\varepsilon)$ of the form (6.24), we see from Eq. (6.25) that $\delta_{\boldsymbol{\xi}}\, c_{mn}$ is simply obtained by projecting the first term on the R.H.S. of (6.25). Remembering that $\mathbf{a}_2 \propto m_1$ and $\mathbf{a}_1 \propto m_2$, we see that the latter term contains the factor $(m_1^2 - m_2^2)\, m_1\, m_2 = m_1^3\, m_2 - m_1\, m_2^3$. Therefore, without doing any further calculation, we see that the shifts $\delta_\xi\, c_{mn}$ have the special properties: $\delta_{\boldsymbol{\xi}}\, c_{22} = 0$ and $\delta_{\boldsymbol{\xi}}\, c_{31} + \delta_{\boldsymbol{\xi}}\, c_{13} = 0$. In other words, a shift of the world-lines of the type (6.24) leaves invariant both c_{22} and the combination $c_{31} + c_{13}$ (as well therefore as the combination $c_{31} + c_{13} - c_{22}$ considered above). As a consequence, we can compute without effort from our previous regularized (but unrenormalized) dim. reg. results (5.24) the following two combinations of the $\xi^i(\varepsilon)$-*renormalized* static coefficients:

$$
\begin{aligned}
c_{22}^{\mathrm{ren}} &= c_{22}^{\mathrm{dr}} = 175 - \frac{41}{16}\,\pi^2\,, &\text{(6.32a)}\\
c_{31}^{\mathrm{ren}} + c_{13}^{\mathrm{ren}} - c_{22}^{\mathrm{ren}} &= c_{31}^{\mathrm{dr}} + c_{13}^{\mathrm{dr}} - c_{22}^{\mathrm{dr}} = -\frac{7}{4}\,. &\text{(6.32b)}
\end{aligned}
$$

By comparing (6.32a) with Eq. (6.29b) we discover that our present calculation using dimensional regularization in harmonic coordinates necessarily implies that

$$
\omega_s = 0 \iff \lambda = -\frac{1987}{3080}\,. \tag{6.33}
$$

This nicely confirms the previous determination of ω_s by a dim. reg. calculation in ADM-type coordinates [35]. We think that our present harmonic-coordinates dim. reg. result calculation is important in proving the consistency of dimensional regularization, and thereby

in confirming the physical significance of the result (6.33). A recent calculation [38, 39] has also confirmed independently the result (6.33) by means of a completely different method based on surface integrals, and aimed at describing compact (strongly-gravitating) objects.

By comparing (6.32b) with (6.30), we further see that

$$r_1' = r_2' \quad \text{[in the case of the dim. reg. shift (6.24)]} \,. \tag{6.34}$$

Contrary to (6.33) which represents the determination of a physical parameter (having an invariant meaning), the result (6.34) has no invariant physical significance. Eq. (6.34) is simply a consequence of our particular choice for the shift vector (6.24), in which we assumed that $e(d)$ is a purely numerical coefficient, independent on any properties indexed by the particles' labels 1 and 2. In summary the particular shift (6.24) yields some equations of motion which are physically equivalent to a subclass of the general equations of motion considered in [22], characterized by the constraint (6.34).

Next we relate the common length scale (6.34) to the basic length scale ℓ_0 entering dimensional regularization. For doing this we need to fully specify the value of the shift, *i.e.*, to choose a specific coefficient $e(d) = e(3) + \varepsilon\, e'(3) + \mathcal{O}(\varepsilon^2)$ in Eq. (6.24). We already know from Eq. (6.10) that the coefficient $e(d)$ in Eq. (6.24) must tend to 11/3 when $d \to 3$, if the ξ-shift is to remove the poles in the bulk metric. As in quantum field theory (QFT) we could then define the *Minimal Subtraction* (MS) shift as

$$\xi^i_{a\mathrm{MS}} \equiv \frac{11}{3\varepsilon}\,\frac{G_N^2\, m_a^2}{c^6}\,(a_a^i)^{d=3} \,. \tag{6.35}$$

However, as is well-known in QFT, such a MS subtraction has the unpleasant feature of leaving some logarithms of π and the Euler constant in the renormalized results. These numbers come from the expansion of the Gamma function and the associated dimension-dependent powers of π entering the d-dimensional Green function. In our context, these numbers showed up in Eq. (5.24) in the guise of the combination

$$\ln(\overline{q}) \equiv \ln(4\pi e^C) = C + \ln(4\pi) \,. \tag{6.36}$$

Like in QFT, this leads us to consider the following *modified minimal subtraction* ($\overline{\mathrm{MS}}$) shift,

$$\xi^i_{a\overline{\mathrm{MS}}} \equiv \frac{11}{3\varepsilon}\,\frac{G_N^2\,\tilde{k}^2\, m_a^2}{c^6}\,a_a^i \,, \tag{6.37}$$

which differs from the MS shift (6.35) by the explicit factor of $\tilde{k}^2$ it contains, and by the use of the d-dimensional (Newtonian) acceleration given by Eq. (6.17). The inclusion of two explicit powers of $\tilde{k}$ in the coefficient $e(d)$ entering Eq. (6.24), *i.e.* the definition $e_{\overline{\mathrm{MS}}}(d) = \frac{11}{3}\,\tilde{k}^2$, means, when remembering that a_a^i, Eq. (6.17), contains one power of $\tilde{k}$, that the static terms in Eq. (6.25) will have four powers of $\tilde{k}$ and the kinetic terms three. The overall factor $\tilde{k}^4$ in the static terms is natural because these terms are of order G^4 and the $\mathbf{x}$-space gravitational propagator in d dimensions always includes the combination $G\,\tilde{k}\,|\mathbf{x}-\mathbf{x}'|^{2-d}$. Finally, using the fact that the expansion of $\tilde{k}(d)$ near $d = 3$ reads

$$\tilde{k}(d) \equiv \frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)}{\pi^{\frac{1+\varepsilon}{2}}} = 1 - \frac{1}{2}\,\varepsilon \ln \overline{q} + \mathcal{O}(\varepsilon^2) \,, \tag{6.38}$$

it is easy to see that the $\overline{\rm MS}$ shift defined by (6.37) will cancel the $\ln\overline{q}$ in the bare dim. reg. results of Eq. (5.24). Finally, we find that the evaluation of Eq. (6.25) for the specific $\overline{\rm MS}$ shift, given by $e_{\overline{\rm MS}}(d) = \frac{11}{3}\,\tilde{k}^2$, yields for the $\overline{\rm MS}$-renormalized static coefficients

$$\begin{aligned} c_{31}^{\overline{\rm MS}} &= \frac{44}{3}\ln\left(\frac{r_{12}}{\ell_0}\right) - \frac{35}{36}\,, & (6.39a)\\ c_{22}^{\overline{\rm MS}} &= 175 - \frac{41}{16}\,\pi^2\,, & (6.39b)\\ c_{13}^{\overline{\rm MS}} &= -\frac{44}{3}\ln\left(\frac{r_{12}}{\ell_0}\right) + \frac{1568}{9} - \frac{41}{16}\,\pi^2\,, & (6.39c)\end{aligned}$$

where the reader can note that the $\ln(r_{12}^2)$ entering the bare dim. reg. result (5.24) have been transformed into $\ln(r_{12}/\ell_0)$ through the ε-expansion of the factor $r_{12}^{-5-2\varepsilon}$ present in $\delta_{\boldsymbol{\xi}}\, a_1^i$.

We already discussed above the comparison of two simple combinations of the dim. reg. results (6.39) with the Hadamard-regularization results (6.28). It is easy to see that the remaining independent combination, say (6.39a), is fully consistent with its counterpart (6.28a), and allows one to relate the basic renormalization length scale ℓ_0 entering dim. reg. to the common length scale (6.34) entering the general equations of motion of [22]:

$$\ln\left(\frac{r'_1}{\ell_0}\right) = \ln\left(\frac{r'_2}{\ell_0}\right) = -\frac{327}{3080} \qquad (\text{for the } \overline{\rm MS} \text{ renormalization})\,. \tag{6.40}$$

Evidently, the precise values one gets for r'_1 and r'_2 depend on the precise choice of the compensating shift.

Let us now remark that in fact one can recover exactly, provided of course that the crucial result (6.33) holds, the general "dissymmetric" class of equations of motion of [22], *i.e.*, the general parametric result (6.28) or (6.29) with $r'_1 \neq r'_2$. For this purpose it suffices to consider a slightly more general shift than the one assumed in the $\overline{\rm MS}$ regularization. Namely, consider a shift of the same form as (6.24), but in which one allows the d-dependent coefficient $e(d)$ to depend on the label of the particle in question, that is

$$\xi_a^i = \frac{e_a(d)}{\varepsilon}\,\frac{G_N^2\,m_a^2}{c^6}\,a_a^i\,, \tag{6.41}$$

where now $e_1(d)$ and $e_2(d)$ are allowed to be different from each other. The most general way of parametrizing such dissymmetric $e_a(d)$ [however constrained by $e_a(3) = 11/3$] is

$$e_a(d) = \frac{11}{3}\tilde{k}^2\left[1 - 2\varepsilon\rho_a + \mathcal{O}(\varepsilon^2)\right]\,, \tag{6.42}$$

with two independent numerical coefficients ρ_a. It is then easily checked that the shift (6.41) defined by the particular choice

$$\rho_a = \ln\left(\frac{r'_a}{\ell_0}\right) + \frac{327}{3080}\,, \tag{6.43}$$

transforms the dim. reg. equations of motion into the general (r'_a-dependent) family of solutions obtained in [22]. If we suppose that the constraints (6.40) hold, then $\rho_a = 0$ and

we recover the shift assumed in the $\overline{\rm MS}$ regularization. On the other hand, note than one can reach even more general classes of renormalized harmonic equations of motion in dim. reg. (as one could have also done in Hadamard regularization). Indeed, we could use the freedom indicated in Eq. (6.15) of adding *arbitrary* finite parts to the shifts. Anyway, the result that the shift (6.41)-(6.43) gives equivalence between the dim. reg. and the extended-Hadamard 3PN accelerations [we check in Section VI D below that the kinetic terms work also], constitutes the main result of the present paper (Theorem 2 in the Introduction).

The two approaches we have discussed here are of course equivalent: choosing some dim. reg. basic length scale ℓ_0 and some specific, simplifying dim. reg. shift (such as the $\overline{\rm MS}$ one), and then determining the values of the scales r'_a for which the dim. reg. results match with the Hadamard reg. ones; or arbitrarily choosing some Hadamard scales r'_a and then determining the corresponding general dissymmetric dim. reg. shift (6.41)-(6.43), in terms of the chosen r'_a's. What is important is that we have checked that the *three* renormalized dim. reg. static coefficients (5.24) are fully compatible with the *three* extended-Hadamard reg. static coefficients (6.28) or (6.29), and that their comparison yields *one and only one* physical result, namely: $\lambda = -\frac{1987}{3080}$.

D. Kinetic terms and check of the consistency of dimensional regularization

Up to now we have verified the following aspects of the consistency of a dim. reg. treatment of the 3PN dynamics of two point particles: (1) consistency between the shift (6.10) needed to renormalize the bulk metric and the shift (6.37) [or (6.41)-(6.42)] needed to renormalize the equations of motion; (2) consistency between the three finite, renormalized dim. reg. static coefficients (6.39) and the general three-dimensional ones (6.28), [22]; and (3) consistency between the present dim. reg. value of λ and the previously derived dim. reg. value of ω_s in the ADM-Hamiltonian [35]. It remains, however, to check that the velocity-dependent terms in the renormalized dim. reg. equations of motion do agree with their analogs in the harmonic-coordinates equations of motion of [22]. This will in particular prove that the dim. reg. equations of motion are Lorentz invariant.

In the notation of Eq. (5.6) above, we need to consider the values of the velocity-dependent coefficients $c_{21}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12})$, $c'_{21}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12})$, $c''_{21}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12})$, $c_{03}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12})$, $c'_{03}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12})$, and $c''_{03}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}_{12})$. In Ref. [22], they were shown to take the following parametric forms, which actually depend only on the regularization scale r'_1 but not on r'_2 nor λ:

$$\begin{aligned}
c_{21}^{\rm BF}(r'_1) &= -22\left[v_{12}^2 - 5(n_{12}v_{12})^2\right]\ln\left(\frac{r_{12}}{r'_1}\right) + \frac{48197}{840}\,v_1^2 - \frac{36227}{420}(v_1v_2) + \frac{36227}{840}\,v_2^2 \\
&\quad -\frac{45887}{168}(n_{12}v_1)^2 + \frac{24025}{42}(n_{12}v_1)(n_{12}v_2) - \frac{10469}{42}(n_{12}v_2)^2\,, && \text{(6.44a)}\\
c_{21}'^{\rm BF}(r'_1) = c_{21}''^{\rm BF}(r'_1) &= -44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right) + \frac{31397}{420}(n_{12}v_1) - \frac{36227}{420}(n_{12}v_2)\,, && \text{(6.44b)}\\
c_{03}^{\rm BF} &= 18(v_1v_2) - 9\,v_2^2 - (n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{43}{2}(n_{12}v_2)^2\,, && \text{(6.44c)}\\
c_{03}'^{\rm BF} = c_{03}''^{\rm BF} &= 4(n_{12}v_1) + 5(n_{12}v_2)\,. && \text{(6.44d)}
\end{aligned}$$

As explained in Eq. (5.1), the dim. reg. expressions of these coefficients can be computed as the sum of pure Hadamard-Schwartz (pHS) contributions, $c_{mn}^{\rm pHS}$ and $c_{mn}'^{\rm pHS}$, and the "dr −

pHS" differences, $\mathcal{D}c_{mn}$, $\mathcal{D}c'_{mn}$. The calculation of the pHS contributions has been explained in Section III D above, and we get the following results from Eqs. (3.55) and (6.44):

$$\begin{aligned}
c_{21}^{\text{pHS}} &= -22\left[v_{12}^2 - 5(n_{12}v_{12})^2\right]\ln\left(\frac{r_{12}}{r'_1}\right) + \frac{10639}{168}\,v_1^2 - \frac{5879}{60}(v_1v_2) + \frac{5843}{120}\,v_2^2 \\
&\quad -\frac{50885}{168}(n_{12}v_1)^2 + \frac{1892}{3}(n_{12}v_1)(n_{12}v_2) - \frac{3325}{12}(n_{12}v_2)^2\,, \qquad (6.45\text{a}) \\
c_{21}^{\prime\text{pHS}} &= -44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right) + \frac{7279}{84}(n_{12}v_1) - \frac{5879}{60}(n_{12}v_2)\,, \qquad (6.45\text{b}) \\
c_{21}^{\prime\prime\text{pHS}} &= -44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right) + \frac{5189}{60}(n_{12}v_1) - \frac{5843}{60}(n_{12}v_2)\,, \qquad (6.45\text{c}) \\
c_{03}^{\text{pHS}} &= 18(v_1v_2) - \frac{64}{7}\,v_2^2 - (n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{311}{14}(n_{12}v_2)^2\,, \qquad (6.45\text{d}) \\
c_{03}^{\prime\text{pHS}} &= 4(n_{12}v_1) + 5(n_{12}v_2)\,, \qquad (6.45\text{e}) \\
c_{03}^{\prime\prime\text{pHS}} &= 4(n_{12}v_1) + \frac{37}{7}(n_{12}v_2)\,. \qquad (6.45\text{f})
\end{aligned}$$

Secondly, the method explained in Sections IV B and V C for computing the differences $\mathcal{D}\mathbf{a}_a$ is found (after doing calculations similar to those reported in the Tables of "static" contributions above) to lead to

$$\begin{aligned}
\mathcal{D}c_{21} &= \left[\frac{11}{\varepsilon} - \frac{33}{2}\ln\left(\overline{q}\,r_1^{\prime 4/3}r_{12}^{2/3}\right)\right]\left[v_{12}^2 - 5(n_{12}v_{12})^2\right] + \frac{499}{42}\,v_1^2 - \frac{359}{15}(v_1v_2) \\
&\quad +\frac{184}{15}\,v_2^2 - \frac{2957}{42}(n_{12}v_1)^2 + \frac{425}{3}(n_{12}v_1)(n_{12}v_2) - \frac{217}{3}(n_{12}v_2)^2 + \mathcal{O}(\varepsilon)\,, \qquad (6.46\text{a}) \\
\mathcal{D}c'_{21} &= \left[\frac{22}{\varepsilon} - 33\ln\left(\overline{q}\,r_1^{\prime 4/3}r_{12}^{2/3}\right)\right](n_{12}v_{12}) + \frac{499}{21}(n_{12}v_1) - \frac{359}{15}(n_{12}v_2) + \mathcal{O}(\varepsilon)\,, \qquad (6.46\text{b}) \\
\mathcal{D}c''_{21} &= \left[\frac{22}{\varepsilon} - 33\ln\left(\overline{q}\,r_1^{\prime 4/3}r_{12}^{2/3}\right)\right](n_{12}v_{12}) + \frac{359}{15}(n_{12}v_1) - \frac{368}{15}(n_{12}v_2) + \mathcal{O}(\varepsilon)\,, \qquad (6.46\text{c}) \\
\mathcal{D}c_{03} &= \frac{1}{7}\,v_2^2 - \frac{5}{7}(n_{12}v_2)^2\,, \qquad (6.46\text{d}) \\
\mathcal{D}c'_{03} &= 0\,, \qquad (6.46\text{e}) \\
\mathcal{D}c''_{03} &= -\frac{2}{7}(n_{12}v_2)\,. \qquad (6.46\text{f})
\end{aligned}$$

Together with Eqs. (5.23) above, these equations give the full difference between the dimensionally regularized and the pure Hadamard accelerations, and they constitute the main new input of the present work. The bare dim. reg. results, $c_{mn}^{\text{dr}} = c_{mn}^{\text{pHS}} + \mathcal{D}c_{mn}$, read therefore

$$\begin{aligned}
c_{21}^{\text{dr}} &= \left[\frac{11}{\varepsilon} - \frac{33}{2}\ln\left(\overline{q}\,r_{12}^2\right)\right]\left[v_{12}^2 - 5(n_{12}v_{12})^2\right] + \frac{1805}{24}\,v_1^2 - \frac{1463}{12}(v_1v_2) \\
&\quad +\frac{1463}{24}\,v_2^2 - \frac{8959}{24}(n_{12}v_1)^2 + \frac{2317}{3}(n_{12}v_1)(n_{12}v_2) - \frac{4193}{12}(n_{12}v_2)^2\,, \qquad (6.47\text{a}) \\
c_{21}^{\prime\text{dr}} = c_{21}^{\prime\prime\text{dr}} &= \left[\frac{22}{\varepsilon} - 33\ln\left(\overline{q}\,r_{12}^2\right)\right](n_{12}v_{12}) + \frac{1325}{12}(n_{12}v_1) - \frac{1463}{12}(n_{12}v_2)\,, \qquad (6.47\text{b}) \\
c_{03}^{\text{dr}} &= 18(v_1v_2) - 9\,v_2^2 - (n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{43}{2}(n_{12}v_2)^2\,, \qquad (6.47\text{c})
\end{aligned}$$

$$c'^{\rm dr}_{03} = c''^{\rm dr}_{03} \; = \; 4(n_{12}v_1) + 5(n_{12}v_2)\,. \tag{6.47d}$$

Note that in the final result the equality between c'_{mn} and c''_{mn}, *i.e.*, between B' and B'' in Eq. (5.3), is recovered. The bare dim. reg. kinetic coefficients (6.47) contain poles $\propto 1/\varepsilon$ but do not depend anymore of the arbitrary Hadamard regularization scale r'_1 which appeared in (6.45). As in the case discussed above of the static coefficients the previous kinetic coefficients do not involve any adimensionalizing length scales in the logarithms of r_{12} they contain. This is consistent with the fact that it is the combinations $\ell_0^{3\varepsilon}\, c^{\rm dr}_{mn}$ and $\ell_0^{3\varepsilon}\, c'^{\rm dr}_{mn}$ which have the same physical dimension as their $d=3$ counterparts.

Finally, given a specific choice of shift, say the $\overline{\rm MS}$ one, Eq. (6.37), the *renormalized* kinetic coefficients are obtained by adding to (6.47) the velocity-dependent part of the effect of the shift, *i.e.*, the second term on the R.H.S. of Eq. (6.25) [with, say, $e^{\overline{\rm MS}}(d) = (11/3)\,\tilde{k}^2$]. Our final ($\overline{\rm MS}$) results for the renormalized kinetic coefficients are found to be

$$\begin{aligned}
c^{\overline{\rm MS}}_{21} \; &= \; -22\left[v_{12}^2 - 5(n_{12}v_{12})^2\right]\ln\left(\frac{r_{12}}{\ell_0}\right) + \frac{1321}{24}\,v_1^2 - \frac{979}{12}(v_1v_2) \\
&\quad + \frac{979}{24}\,v_2^2 - \frac{6275}{24}(n_{12}v_1)^2 + \frac{1646}{3}(n_{12}v_1)(n_{12}v_2) - \frac{2851}{12}(n_{12}v_2)^2\,, \qquad (6.48\text{a}) \\
c'^{\overline{\rm MS}}_{21} = c''^{\overline{\rm MS}}_{21} \; &= \; -44(n_{12}v_{12})\ln\left(\frac{r_{12}}{\ell_0}\right) + \frac{841}{12}(n_{12}v_1) - \frac{979}{12}(n_{12}v_2)\,, \qquad (6.48\text{b}) \\
c^{\overline{\rm MS}}_{03} \; &= \; 18(v_1v_2) - 9\,v_2^2 - (n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{43}{2}(n_{12}v_2)^2\,, \qquad (6.48\text{c}) \\
c'^{\overline{\rm MS}}_{03} = c''^{\overline{\rm MS}}_{03} \; &= \; 4(n_{12}v_1) + 5(n_{12}v_2)\,. \qquad (6.48\text{d})
\end{aligned}$$

When comparing these results with the ones of Ref. [22], Eqs. (6.44) above, one remarkably finds that our previously derived link (6.40) is necessary and sufficient for ensuring the full compatibility between the renormalized dim. reg. results and the corresponding Hadamard reg. ones. Note that the rational coefficients entering the dim. reg. results are often simpler than the coefficients entering the equations of motion of [22].

The results Eq. (6.48) complete our check of the full consistency of the dim. reg. evaluation of the 3PN equations of motion, and the proof of Theorems 1 and 2 stated in the Introduction.

VII. CONCLUSIONS

We have used dimensional regularization (*i.e.* analytic continuation in the spatial dimension d) to determine the spacetime metric and the equations of motion (EOM) in *harmonic coordinates*, of two, gravitationally interacting, point masses, at the third post-Newtonian (3PN) order of General Relativity. Our starting point consisted in writing the 3PN-accurate metric $g_{\mu\nu}(x)$ in terms of a certain number of "elementary potentials" V, V_i, $\hat{W}_{ij}$, $\cdots$, satisfying a hierarchy of inhomogeneous d'Alembert equations of the form $\Box$(potential) = source. The sources of the latter equations contain both "compact" terms, *i.e.*, in the present case *contact* terms of the form $\mathcal{F}[V(\mathbf{x}), V_i(\mathbf{x}), \hat{W}_{ij}(\mathbf{x}), \cdots]\delta^{(d)}(\mathbf{x}-\mathbf{y}_1)$, and nonlinearly generated "non compact" terms of the typical form, say, ∂(potential)∂(potential). This representation of the 3PN metric, as well as the associated iterative way of solving for the potentials [using

the time-symmetric Green's function $\Box^{-1} = \Delta^{-1} + c^{-2}\Delta^{-2}\partial_t^2 + \mathcal{O}(c^{-4})$] is a direct generalization of the one used in Ref. [22]. However, it has been crucial for our work to determine (in Section II) the dependence upon the dimension d of the coefficients appearing in this representation, as well as the d-dependence of the kernels expressing the operators Δ^{-1} and Δ^{-2} in $\mathbf{x}$-space.

By studying the structure of the iterative solution for the metric, and that of the corresponding EOM (which are conveniently pictured by means of diagrams, see Figs. 1-8), we determined, in the form of a Laurent expansion in $\varepsilon \equiv d-3$, the pole part of the metric $g_{\mu\nu}(x)$, and the pole and finite parts of the EOM, namely $\mathbf{a}_a = A_a(\mathbf{y}_b, \mathbf{v}_b)$ where $a, b = 1, 2$ and $\mathbf{v}_b \equiv d\mathbf{y}_b/dt$. [See, however, Appendix C where the basic quadratically non-linear kernel $\mathrm{g}(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$ is computed in any d dimensions, not necessarily close to 3.] Our calculations relied heavily on previous work in $d = 3$ [22], and were technically implemented in two steps (at least for the determination of the EOM, which are more delicate; the determination of the pole part of the metric uses only the second step).

(i) A first step consisted of subtracting from the final, published results for the EOM [22], seven contributions that were specific consequences of the use of an extension of the Hadamard regularization method [23, 24] (which included an extension of the Schwartz notion of distributional derivative). The result of this first step is referred to as the "pure Hadamard-Schwartz" evaluation of the EOM.

(ii) The second step was the evaluation of the *difference* between the dimensional regularization of each contribution to the EOM (written in terms of the iterative solutions for the various potentials V, V_i, $\hat{W}_{ij}$, $\cdots$), and the corresponding "pure Hadamard-Schwartz" contribution obtained in the first step. This difference is obtained, similarly to the method used in [35], by splitting the d-dimensional integral into several pieces, and by carefully analyzing the terms due to the neighborhoods of the two singular points $\mathbf{y}_1$, $\mathbf{y}_2$ (including possible d-dimensional distributional contributions).

Concerning the "bulk" metric $g_{\mu\nu}(x)$, at a field point away from the singular particle world-lines $\mathbf{y}_a(t)$, we derived only the pole part, that is the coefficient of $1/\varepsilon$ in the Laurent expansion of $g_{\mu\nu}(x;\varepsilon)$. We found that at the 3PN order only the time-time component of the metric $g_{00}(x)$ contained a pole [see Eq. (6.9)]. For the EOM we derived both the pole part and the finite part, *i.e.* $\mathbf{a}_a \sim \varepsilon^{-1} + \varepsilon^0 + \mathcal{O}(\varepsilon)$. The parts of the EOM for which the regularization was delicate are given by the nine coefficients c_{31}, $\cdots$, c''_{03} defined in Eq. (5.6). Our complete results for the dimensionally-regularized values of these nine "delicate" coefficients are given in Eqs. (5.24) and (6.47).

We proved that the pole parts of both the metric and the EOM can be "renormalized away" by suitable *shifts of the world-lines* of the form $\mathbf{y}_a^{\mathrm{bare}} = \mathbf{y}_a^{\mathrm{ren}} + \boldsymbol{\xi}_a(\varepsilon)$, where $\mathbf{y}_a^{\mathrm{bare}}$ is the original world-line on which are initially concentrated the δ-function sources representing the point masses, where the shifts $\boldsymbol{\xi}_a(\varepsilon) \sim \varepsilon^{-1} + \varepsilon^0 + \mathcal{O}(\varepsilon)$ are of the 3PN order, and where the EOM of the renormalized world-line $\mathbf{y}_a^{\mathrm{ren}}$ is *finite* as $\varepsilon \to 0$. The general form of the needed shifts is given by Eq. (6.15) with (6.10). The renormalized EOM corresponding to the "modified Minimal Subtraction" scheme (6.37) are given by Eqs. (6.39) and (6.48).

The finite renormalized 3PN-accurate EOM obtained by using the general (two-parameter) renormalization shift (6.41)-(6.43) were shown to be *equivalent* to the final (three-parameter) EOM of [22] if and only if the Hadamard-undetermined dimensionless parameter λ which entered the latter equations takes the unique value $\lambda = -\frac{1987}{3080}$. This value is in agreement with the result of a previous dimensional-regularization determination of the Arnowitt-Deser-Misner Hamiltonian (in ADM-like coordinates) [35], which led to the

unique determination of the ADM analogue of λ, namely $\omega_s = 0$. The value for λ or ω_s is also in agreement with the recent work [38, 39] which derived the 3PN equations of motion in harmonic gauge using a surface-integral approach. Our result provides an important check of the consistency of dimensional regularization because our calculations are very different from the ones of [35], notably we use a different coordinate system and a different method for iterating Einstein's field equations. However, the applicability of our general approach to higher post-Newtonian orders remains unexplored.

Finally, the present work opens the way to a dimensional-regularization determination of the several unknown dimensionless parameters that were shown to enter the Hadamard-regularization of the 3PN binary's energy flux (in harmonic coordinates) [27, 28]. The completion of the 3PN energy flux is urgent in view of its importance in determining the gravitational waveforms emitted by inspiralling black hole binaries, which are primary targets for the international network of interferometric gravitational wave detectors LIGO/VIRGO/GEO.

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APPENDIX A: THE d-DIMENSIONAL METRIC AND GEODESIC EQUATION

We give in this Appendix several expanded expressions which are too lengthy to be included in the body of the article. The expanded form of the metric (2.11) is easier to compare with the literature, and notably with Eqs. (3.24) of Ref. [22]:

$$\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}\left[V^2 + 2\left(\frac{d-3}{d-2}\right)K\right] + \frac{8}{c^6}\left[\hat{X} + V_iV_i + \frac{1}{6}V^3 + \left(\frac{d-3}{d-2}\right)VK\right] \\ &\quad + \frac{32}{c^8}\left[\hat{T} - \frac{1}{2}V\hat{X} + \hat{R}_iV_i - \frac{1}{2}VV_iV_i - \frac{1}{48}V^4 + \frac{1}{4}\left(\frac{d-3}{d-2}\right)KV^2 - \frac{1}{4}\left(\frac{d-3}{d-2}\right)^2K^2\right] \\ &\quad + \mathcal{O}\left(\frac{1}{c^{10}}\right), \end{aligned} \tag{A1a}$$

$$\begin{aligned} g_{0i} &= -\frac{4}{c^3}V_i - \frac{8}{c^5}\left[\hat{R}_i - \frac{1}{2}\left(\frac{d-3}{d-2}\right)VV_i\right] - \frac{16}{c^7}\Bigg[\hat{Y}_i + \frac{1}{2}\hat{W}_{ij}V_j \\ &\quad + \frac{1}{4}\left(1 + \frac{1}{(d-2)^2}\right)V^2V_i - \frac{1}{2}\left(\frac{d-3}{d-2}\right)V\hat{R}_i + \frac{1}{2}\left(\frac{d-3}{d-2}\right)^2KV_i\Bigg] + \mathcal{O}\left(\frac{1}{c^9}\right), \end{aligned} \tag{A1b}$$

$$\begin{aligned} g_{ij} &= \delta_{ij}\Bigg\{1 + \frac{2}{(d-2)c^2}V + \frac{2}{(d-2)^2c^4}\left[V^2 - 2(d-3)K\right] \\ &\quad + \frac{8}{c^6}\left[\frac{\hat{X}}{d-2} + \frac{V_kV_k}{d-2} + \frac{V^3}{6(d-2)^3} - \frac{(d-3)}{(d-2)^3}VK\right]\Bigg\} \end{aligned}$$

$$+\frac{4}{c^4}\,\hat{W}_{ij}+\frac{16}{c^6}\left[\hat{Z}_{ij}+\frac{V\hat{W}_{ij}}{2(d-2)}-V_iV_j\right]+\mathcal{O}\left(\frac{1}{c^8}\right). \tag{A1c}$$

The inverse metric is such that $g_{\mu\nu}g^{\nu\rho}=\delta_\mu^\rho$ in $d+1$ space-time dimensions. In terms of the modified Newtonian potential $\mathcal{V}$ defined in Eq. (2.10) above, it reads:

$$g^{00} = -e^{2\mathcal{V}/c^2}\left(1-\frac{8V_iV_i}{c^6}-\frac{32\hat{R}_iV_i}{c^8}\right)+\mathcal{O}\left(\frac{1}{c^{10}}\right), \tag{A2a}$$

$$g^{0i} = -e^{\frac{(d-3)\mathcal{V}}{(d-2)c^2}}\left\{\frac{4V_i}{c^3}\left[1+\frac{1}{2}\left(\frac{d-1}{d-2}\,\frac{V}{c^2}\right)^2\right]+\frac{8\hat{R}_i}{c^5}+\frac{16}{c^7}\left[\hat{Y}_i-\frac{1}{2}\hat{W}_{ij}V_j\right]\right\}+\mathcal{O}\left(\frac{1}{c^9}\right), \tag{A2b}$$

$$g^{ij} = e^{-\frac{2\mathcal{V}}{(d-2)c^2}}\left\{\delta_{ij}-\frac{4}{c^4}\hat{W}_{ij}-\frac{16}{c^6}\left[\hat{Z}_{ij}+\frac{1}{2(d-2)}\,\delta_{ij}V_kV_k\right]\right\}+\mathcal{O}\left(\frac{1}{c^8}\right). \tag{A2c}$$

Note the change of signs in the exponentials [with respect to the covariant metric (2.11)], in front of $\hat{W}_{ij}V_j$ in Eq. (A2b), as well as for the $\mathcal{O}(1/c^4)$ and $\mathcal{O}(1/c^6)$ terms in Eq. (A2c). Note also that the V_iV_j contribution to g_{ij} has disappeared in the inverse spatial metric g^{ij}. The full post-Newtonian expansion of this inverse metric reads:

$$\begin{aligned}
g^{00} &= -1-\frac{2}{c^2}\,V-\frac{2}{c^4}\left[V^2-2\left(\frac{d-3}{d-2}\right)K\right]-\frac{8}{c^6}\left[\hat{X}-V_iV_i+\frac{V^3}{6}-\left(\frac{d-3}{d-2}\right)VK\right]\\
&\quad-\frac{32}{c^8}\left[\hat{T}+\frac{1}{2}V\hat{X}-\hat{R}_iV_i-\frac{1}{2}VV_iV_i+\frac{V^4}{48}-\frac{1}{4}\left(\frac{d-3}{d-2}\right)KV^2+\frac{1}{4}\left(\frac{d-3}{d-2}\right)^2K^2\right]\\
&\quad+\mathcal{O}\left(\frac{1}{c^{10}}\right),
\end{aligned} \tag{A3a}$$

$$\begin{aligned}
g^{0i} &= -\frac{4}{c^3}\,V_i-\frac{8}{c^5}\left[\hat{R}_i+\frac{1}{2}\left(\frac{d-3}{d-2}\right)VV_i\right]-\frac{16}{c^7}\left[\hat{Y}_i-\frac{1}{2}\,\hat{W}_{ij}V_j\right.\\
&\quad\left.+\frac{1}{4}\left(1+\frac{1}{(d-2)^2}\right)V^2V_i+\frac{1}{2}\left(\frac{d-3}{d-2}\right)V\hat{R}_i-\frac{1}{2}\left(\frac{d-3}{d-2}\right)^2KV_i\right]+\mathcal{O}\left(\frac{1}{c^9}\right),
\end{aligned} \tag{A3b}$$

$$\begin{aligned}
g^{ij} &= \delta_{ij}\left\{1-\frac{2}{(d-2)}\,\frac{V}{c^2}+\frac{2}{(d-2)^2c^4}\left[V^2+2(d-3)K\right]\right.\\
&\quad\left.-\frac{8}{c^6}\left[\frac{\hat{X}}{d-2}+\frac{V_kV_k}{d-2}+\frac{V^3}{6(d-2)^3}+\frac{d-3}{(d-2)^3}\,VK\right]\right\}\\
&\quad-\frac{4}{c^4}\,\hat{W}_{ij}-\frac{16}{c^6}\left[\hat{Z}_{ij}-\frac{1}{2(d-2)}\,V\hat{W}_{ij}\right]+\mathcal{O}\left(\frac{1}{c^8}\right).
\end{aligned} \tag{A3c}$$

The determinant $g\equiv\det g_{\mu\nu}$ of the metric is a useful quantity, notably to compute the "gothic" metric $\mathfrak{g}^{\mu\nu}\equiv\sqrt{-g}\,g^{\mu\nu}$, which is the natural variable when using the harmonic-coordinate system. The simplest way to compute it is to use the exponential form (2.11) of the metric, and to perform a cofactor expansion across both the first line and the first column:

$$\det g_{\mu\nu} = g_{00}\det g_{ij}-\sum_{k=1}^{d}\sum_{l=1}^{d}(-)^{k+l}g_{0k}\,g_{0l}\det(g_{i\neq k\ j\neq l})\,. \tag{A4}$$

Since $g_{ij} = \exp\left[\frac{2\mathcal{V}}{(d-2)c^2}\right] \times [\delta_{ij} + \mathcal{O}(1/c^4)]$, the determinant of the $(d-1)\times(d-1)$ matrix $g_{i\neq k\ j\neq l}$ reads

$$
\begin{aligned}
\det(g_{i\neq k\ j\neq l}) &= e^{\frac{2(d-1)\mathcal{V}}{(d-2)c^2}} \det(\delta_{i\neq k\ j\neq l}) + \mathcal{O}(1/c^4) \\
&= e^{\frac{2(d-1)\mathcal{V}}{(d-2)c^2}} \delta_{kl} + \mathcal{O}(1/c^4)\,. \qquad \text{(A5)}
\end{aligned}
$$

Therefore, the determinant of the full metric is given by

$$
g \equiv \det g_{\mu\nu} = g_{00} \det g_{ij} - e^{\frac{2(d-1)\mathcal{V}}{(d-2)c^2}} (g_{0i})^2 + \mathcal{O}\left(\frac{1}{c^{10}}\right), \qquad \text{(A6)}
$$

where we have used the fact that $g_{0i} = \mathcal{O}(1/c^3)$. Note that this formula suffices to compute g up to order $\mathcal{O}(1/c^8)$ included if one knows the spatial metric g_{ij} up to this same order. At the 3PN order, we get easily

$$
\begin{aligned}
g &= -e^{\frac{4\mathcal{V}}{(d-2)c^2}} \left\{ \left(1 - \frac{8}{c^6} V_i V_i\right) \det\left[\delta_{ij} + \frac{4}{c^4} \hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} - V_i V_j + \frac{1}{2(d-2)} \delta_{ij} V_k V_k\right)\right] \right. \\
&\qquad \left. + \frac{16}{c^6} V_i V_i \right\} + \mathcal{O}\left(\frac{1}{c^8}\right) \\
&= -e^{\frac{4\mathcal{V}}{(d-2)c^2}} \left[1 + \frac{4}{c^4} \hat{W} + \frac{16}{c^6}\left(\hat{Z} + \frac{1}{d-2} V_i V_i\right)\right] + \mathcal{O}\left(\frac{1}{c^8}\right), \qquad \text{(A7)}
\end{aligned}
$$

where we used the expansion $\det(\mathbb{1} + M) = 1 + \mathrm{Tr}\, M + \mathcal{O}(M^2)$, valid for any matrix M whose entries are small with respect to 1. We can now compute the square root of this determinant, and give its full post-Newtonian expansion:

$$
\sqrt{-g} = e^{\frac{2\mathcal{V}}{(d-2)c^2}} \left[1 + \frac{2}{c^4} \hat{W} + \frac{8}{c^6}\left(\hat{Z} + \frac{1}{d-2} V_i V_i\right)\right] + \mathcal{O}\left(\frac{1}{c^8}\right) \qquad \text{(A8a)}
$$

$$
\begin{aligned}
&= 1 + \frac{2}{(d-2)} \frac{V}{c^2} + \frac{2}{c^4}\left[\hat{W} + \frac{V^2}{(d-2)^2} - \frac{2(d-3)}{(d-2)^2} K\right] + \frac{8}{c^6}\left[\hat{Z} + \frac{V_i V_i}{d-2}\right. \\
&\qquad \left. + \frac{\hat{X}}{d-2} + \frac{V\hat{W}}{2(d-2)} + \frac{V^3}{6(d-2)^3} - \frac{d-3}{(d-2)^3} V K\right] + \mathcal{O}\left(\frac{1}{c^8}\right). \qquad \text{(A8b)}
\end{aligned}
$$

The gothic metric $\mathfrak{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu}$ can now be written easily by combining Eqs. (A2) or (A3) with (A8a) or (A8b). We shall not display here the explicit results, since they were not directly useful for the present article. Let us however quote the expression of the Ricci tensor in terms of the gothic metric, in $d+1$ space-time dimensions and in any gauge:

$$
\begin{aligned}
2R_{\mu\nu} &= -\mathfrak{g}_{\mu\alpha}\, \mathfrak{g}^{\alpha\beta}{}_{,\beta\nu} - \mathfrak{g}_{\nu\alpha}\, \mathfrak{g}^{\alpha\beta}{}_{,\beta\mu} \\
&\quad + \left(\mathfrak{g}_{\mu\alpha}\, \mathfrak{g}_{\nu\beta} - \frac{1}{d-1} \mathfrak{g}_{\mu\nu}\, \mathfrak{g}_{\alpha\beta}\right) \left(\mathfrak{g}^{\gamma\delta}\, \mathfrak{g}^{\alpha\beta}{}_{,\gamma\delta} - \mathfrak{g}_{\gamma\delta}\, \mathfrak{g}^{\varepsilon\zeta}\, \mathfrak{g}^{\alpha\gamma}{}_{,\varepsilon}\, \mathfrak{g}^{\beta\delta}{}_{,\zeta} + \mathfrak{g}^{\alpha\beta}{}_{,\gamma}\, \mathfrak{g}^{\gamma\delta}{}_{,\delta}\right) \\
&\quad - \frac{1}{2} \mathfrak{g}_{\alpha\beta}\, \mathfrak{g}_{\gamma\delta} \left(\mathfrak{g}^{\alpha\gamma}{}_{,\mu}\, \mathfrak{g}^{\beta\delta}{}_{,\nu} - \frac{1}{d-1} \mathfrak{g}^{\alpha\beta}{}_{,\mu}\, \mathfrak{g}^{\gamma\delta}{}_{,\nu}\right) - \mathfrak{g}_{\mu\alpha}\, \mathfrak{g}_{\nu\beta}\, \mathfrak{g}^{\alpha\gamma}{}_{,\delta}\, \mathfrak{g}^{\beta\delta}{}_{,\gamma} \\
&\quad + \mathfrak{g}_{\alpha\beta} \left(\mathfrak{g}_{\mu\gamma}\, \mathfrak{g}^{\beta\delta}{}_{,\nu} + \mathfrak{g}_{\nu\gamma}\, \mathfrak{g}^{\beta\delta}{}_{,\mu}\right) \mathfrak{g}^{\alpha\gamma}{}_{,\delta}\,. \qquad \text{(A9)}
\end{aligned}
$$

As usual, a comma denotes partial derivation, and $\mathfrak{g}_{\mu\nu} \equiv g_{\mu\nu}/\sqrt{-g}$ is the inverse of $\mathfrak{g}^{\mu\nu}$. In terms of the gothic metric, the harmonic gauge condition (2.1) takes a particularly simple form:

$$\mathfrak{g}^{\mu\alpha}{}_{,\alpha} = -\sqrt{-g}\, g^{\alpha\beta}\Gamma^{\lambda}_{\alpha\beta} = 0\,. \tag{A10}$$

This is the reason why this gothic metric can be useful to write the field equations. Note that several terms of Eq. (A9) vanish in this gauge, namely the first two (involving second derivatives) and those proportional to $\mathfrak{g}^{\gamma\delta}{}_{,\delta}$ in the second line. Nevertheless, this expression for $R_{\mu\nu}$ is slightly more complicated than the one we used in Section II above, Eq. (2.2), which does not depend explicitly on the spatial dimension d. It should be noted that many equations given in the book [46] are erroneous when $d \neq 3$ (*i.e.*, when $d+1 = n \neq 4$, in this book's notation), including Eq. (I, 14, 30) in [46] which gives the Ricci tensor in terms of the gothic metric.

Let us end this Appendix by displaying the full expansion of the geodesic equation (2.19), or more precisely of the vectors P^i and F^i, quickly illustrated in Eqs. (2.22). The following expressions are d-dimensional generalizations of Eqs. (3.35) of Ref. [22], and we keep the same writing and order of the terms to ease the comparison. The "linear momentum" P^i reads

$$\begin{aligned}
P^i \;=\;& v^i + \frac{1}{c^2}\left(\frac{1}{2}\,v^2v^i + \frac{d}{d-2}\,Vv^i - 4V_i\right) + \frac{1}{c^4}\left[\frac{3}{8}\,v^4v^i + \frac{3d-2}{2(d-2)}\,Vv^2v^i - 4V_jv^iv^j\right.\\
&\left.-2V_iv^2 + \frac{d^2}{2(d-2)^2}\,V^2v^i - \frac{4}{d-2}\,VV_i + 4\hat{W}_{ij}v^j - 8\hat{R}_i - \frac{2d(d-3)}{(d-2)^2}\,Kv^i\right]\\
&+\frac{1}{c^6}\left[\frac{5}{16}\,v^6v^i + \frac{3(5d-4)}{8(d-2)}\,Vv^4v^i - \frac{3}{2}\,V_iv^4 - 6V_jv^iv^jv^2 + \frac{(3d-2)^2}{4(d-2)^2}\,V^2v^2v^i\right.\\
&+2\hat{W}_{ij}v^jv^2 + 2\hat{W}_{jk}v^iv^jv^k - \frac{2(2d-1)}{d-2}\,VV_iv^2 - \frac{4(2d-1)}{d-2}\,VV_jv^iv^j - 4\hat{R}_iv^2\\
&-8\hat{R}_jv^iv^j + \frac{d^3}{6(d-2)^3}\,V^3v^i + \frac{4d}{d-2}\,V_jV_jv^i + \frac{4d}{d-2}\,\hat{W}_{ij}Vv^j + \frac{4d}{d-2}\,\hat{X}v^i\\
&+16\hat{Z}_{ij}v^j - 2\,\frac{d(d-2)+2}{(d-2)^2}\,V^2V_i - 8\hat{W}_{ij}V^j - \frac{8}{d-2}V\hat{R}_i - 16\hat{Y}_i\\
&\left.-\frac{(3d-2)(d-3)}{(d-2)^2}\,Kv^2v^i - \frac{2d^2(d-3)}{(d-2)^3}\,KVv^i + \frac{8(d-3)}{(d-2)^2}\,KV_i\right] + \mathcal{O}\left(\frac{1}{c^8}\right).
\end{aligned} \tag{A11}$$

This d-dimensional expression actually allows us to understand better some of the numerical coefficients found for $d=3$ in Ref. [22]. For instance, we find that a factor 33 comes from the expression $3(5d-4)$, and that a factor 49 comes from $(3d-2)^2$. The full post-Newtonian expansion of the "force" F^i is given by an even longer formula:

$$\begin{aligned}
F^i \;=\;& \partial_iV + \frac{1}{c^2}\left[-V\partial_iV + \frac{d}{2(d-2)}\,\partial_iVv^2 - 4\partial_iV_jv^j - 2\left(\frac{d-3}{d-2}\right)\partial_iK\right]\\
&+\frac{1}{c^4}\left[\frac{3d-2}{8(d-2)}\,\partial_iVv^4 - 2\partial_iV_jv^jv^2 + \frac{d^2}{2(d-2)^2}\,V\partial_iVv^2 + 2\partial_i\hat{W}_{jk}v^jv^k\right.\\
&-\frac{4}{d-2}\left(V_j\,\partial_iVv^j + V\partial_iV_jv^j\right) - 8\partial_i\hat{R}_jv^j + \frac{1}{2}\,V^2\partial_iV + 8V_j\,\partial_iV_j\\
&\left.+4\partial_i\hat{X} + 2\left(\frac{d-3}{d-2}\right)(K\partial_iV + V\partial_iK) - \frac{d(d-3)}{(d-2)^2}\,\partial_iKv^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{c^6}\left[\frac{1}{16}\left(\frac{5d-4}{d-2}\right)v^6\partial_i V-\frac{3}{2}\,\partial_i V_j v^j v^4+\frac{1}{8}\left(\frac{3d-2}{d-2}\right)^2 V\partial_i V v^4+\partial_i\hat{W}_{jk}v^2 v^j v^k\right.\\
&-2\left(\frac{2d-1}{d-2}\right)V_j\,\partial_i V v^2 v^j-2\left(\frac{2d-1}{d-2}\right)V\partial_i V_j v^2 v^j-4\partial_i\hat{R}_j v^2 v^j\\
&+\frac{1}{4}\left(\frac{d}{d-2}\right)^3 V^2\partial_i V v^2+\frac{4d}{d-2}\,V_j\partial_i V_j v^2+\frac{2d}{d-2}\,\hat{W}_{jk}\,\partial_i V v^j v^k\\
&+\frac{2d}{d-2}\,V\partial_i\hat{W}_{jk}v^j v^k+\frac{2d}{d-2}\,\partial_i\hat{X}v^2+8\partial_i\hat{Z}_{jk}v^j v^k-4\,\frac{d(d-2)+2}{(d-2)^2}\,V_j V\partial_i V v^j\\
&-2\,\frac{d(d-2)+2}{(d-2)^2}\,V^2\partial_i V_j v^j-8V_k\partial_i\hat{W}_{jk}v^j-8\hat{W}_{jk}\partial_i V_k v^j-\frac{8}{d-2}\,\hat{R}_j\partial_i V v^j\\
&-\frac{8}{d-2}\,V\partial_i\hat{R}_j v^j-16\partial_i\hat{Y}_j v^j-\frac{1}{6}\,V^3\partial_i V-4V_j V_j\partial_i V+16\hat{R}_j\partial_i V_j+16V_j\partial_i\hat{R}_j\\
&-8VV_j\partial_i V_j-4\hat{X}\partial_i V-4V\partial_i\hat{X}+16\partial_i\hat{T}-\frac{d^2(d-3)}{(d-2)^3}\,K\partial_i V v^2-2\left(\frac{d-3}{d-2}\right)KV\partial_i V\\
&+\frac{8(d-3)}{(d-2)^2}\,K\partial_i V_j v^j-\frac{(3d-2)(d-3)}{4(d-2)^2}\,\partial_i K v^4-\frac{d^2(d-3)}{(d-2)^3}\,\partial_i K V v^2\\
&\left.-\frac{d-3}{d-2}\,V^2\partial_i K+\frac{8(d-3)}{(d-2)^2}\,\partial_i K V_j v^j-4\left(\frac{d-3}{d-2}\right)^2 K\partial_i K\right]+\mathcal{O}\left(\frac{1}{c^8}\right).
\end{aligned}
\tag{A12}
$$

APPENDIX B: USEFUL FORMULAE IN d DIMENSIONS

This appendix is intended to provide a compendium of (mostly well-known) formulae for working in a space with d dimensions. As usual, though we shall motivate some formulae below by writing some intermediate expressions which make complete sense only when d is a strictly positive integer, our final formulae are to be interpreted, by complex analytic continuation, for a general complex dimension, $d \in \mathbb{C}$. Actually one of the main sources of the power of dimensional regularization is its ability to prove many results by invoking complex analytic continuation in d.

We discuss first the volume of the sphere having $d-1$ dimensions (*i.e.*, embedded into Euclidean d-dimensional space). We separate out the infinitesimal volume element in d dimensions into radial and angular parts,

$$
d^d\mathbf{x} = r^{d-1} dr\, d\Omega_{d-1}\,, \tag{B1}
$$

where $r = |\mathbf{x}|$ denotes the radial variable (*i.e.*, the Euclidean norm of $\mathbf{x} \in \mathbb{R}^d$) and $d\Omega_{d-1}$ is the infinitesimal solid angle sustained by the unit sphere with $d-1$ dimensional surface. To compute the volume of the sphere, $\Omega_{d-1} = \int d\Omega_{d-1}$, one notices that the following d-dimensional integral can be computed both in Cartesian coordinates, where it reduces simply to a Gaussian integral, and also, using (B1), in spherical coordinates:

$$
\int d^d\mathbf{x}\, e^{-r^2} = \left(\int dx\, e^{-x^2}\right)^d = \pi^{\frac{d}{2}} = \Omega_{d-1}\int_0^{+\infty} dr\, r^{d-1} e^{-r^2} = \frac{1}{2}\Omega_{d-1}\Gamma\left(\frac{d}{2}\right), \tag{B2}
$$

where Γ in the last equation denotes the Eulerian function. This leads to the well known result

$$\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\,. \tag{B3}$$

For instance one recovers the standard results $\Omega_2 = 4\pi$ and $\Omega_1 = 2\pi$, but also $\Omega_0 = 2$, which can be interpreted by remarking that the sphere with 0 dimension is actually made of two points. If we parametrize the sphere Ω_{d-1} in $d-1$ dimensions by means of $d-1$ spherical coordinates θ_{d-1}, θ_{d-2}, $\cdots$, which are such that the sphere Ω_{d-2} in $d-2$ dimensions is then parametrized by θ_{d-2}, θ_{d-3}, $\cdots$, and so on for the lower-dimensional spheres, then we find that the differential volume elements on each of the successive spheres obey the recursive relation

$$d\Omega_{d-1} = \left(\sin\theta_{d-1}\right)^{d-2} d\theta_{d-1} d\Omega_{d-2}\,. \tag{B4}$$

Note that this implies

$$\frac{\Omega_{d-1}}{\Omega_{d-2}} = \int_0^\pi d\theta_{d-1}\left(\sin\theta_{d-1}\right)^{d-2} = \int_{-1}^{+1} dx \left(1-x^2\right)^{\frac{d-3}{2}}\,, \tag{B5}$$

which can also be checked directly by using the explicit expression (B3).

Next we consider the Dirac delta-function $\delta^{(d)}(\mathbf{x})$ in d dimensions, which is formally defined, as in ordinary distribution theory [36], by the following linear form acting on the set $\mathcal{D}$ of smooth functions $\in C^\infty(\mathbb{R}^d)$ with compact support: $\forall\varphi\in\mathcal{D}$,

$$<\delta^{(d)},\varphi>\equiv \int d^d\mathbf{x}\,\delta^{(d)}(\mathbf{x})\varphi(\mathbf{x}) = \varphi(\mathbf{0})\,, \tag{B6}$$

where the brackets refer to the action of a distribution on $\varphi\in\mathcal{D}$. Let us now check that the function defined by

$$u \;=\; \tilde{k}\, r^{2-d}\,, \tag{B7a}$$

$$\tilde{k} \;=\; \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}\,, \tag{B7b}$$

[where r is the radial coordinate in d dimensions, such that $r^2 = \sum_{i=1}^d (x^i)^2$] is the "Green's function" of the Poisson operator, namely that it obeys the distributional equation

$$\Delta u = -4\pi\delta^{(d)}(\mathbf{x})\,. \tag{B8}$$

For any $\alpha\in\mathbb{C}$ we have $\Delta r^\alpha = \alpha(\alpha+d-2)\,r^{\alpha-2}$, thus we see that $\Delta u = 0$ in the sense of functions. Let us formally compute its value in the sense of distributions in $\mathbf{x}$-space. [The usual verification of (B8) is done in Fourier space.] We apply the distribution Δu on some test function $\varphi\in\mathcal{D}$, use the definition of the distributional derivative to shift the Laplace operator from u to φ, compute the value of the d-dimensional integral by removing a ball of small radius s surrounding the origin [say $B(s)$], apply the fact that $\Delta u = 0$ in the exterior of $B(s)$, use the Gauss theorem to transform the result into a surface integral, and finally compute that integral by inserting the Taylor expansion of φ around the origin. The proof

of Eq. (B8) is thus summarized in the following steps:

$$\begin{aligned}
< \Delta u, \varphi > &= < u, \Delta\varphi > \\
&= \lim_{s\to 0} \int_{\mathbb{R}^d \setminus B(s)} d^d\mathbf{x}\, u\Delta\varphi \\
&= \lim_{s\to 0} \int_{\mathbb{R}^d \setminus B(s)} d^d\mathbf{x}\, \partial_i \left[u\partial_i\varphi - \partial_i u\, \varphi\right] \\
&= \lim_{s\to 0} \int s^{d-1} d\Omega_{d-1} (-n_i) \left[u\partial_i\varphi - \partial_i u\, \varphi\right] \\
&= \lim_{s\to 0} \int s^{d-1} d\Omega_{d-1} (-n_i) \left[-\tilde{k}\,(2-d) s^{1-d} n_i \varphi(\mathbf{0})\right] \\
&= \Omega_{d-1} \tilde{k}\,(2-d)\varphi(\mathbf{0}) \\
&= -4\pi\varphi(\mathbf{0})\,.
\end{aligned} \tag{B9}$$

In the last step we used the relation between $\tilde{k}$ and the volume of the sphere, which is

$$\tilde{k}\,\Omega_{d-1} = \frac{4\pi}{d-2}\,. \tag{B10}$$

From $u = \tilde{k}\, r^{2-d}$ one can next find the solution v satisfying the equation $\Delta v = u$ (in a distributional sense), namely

$$v = \frac{\tilde{k}\, r^{4-d}}{2(4-d)}\,. \tag{B11}$$

From (B11) we can then define a whole "hierarchy" of higher-order functions $w, \cdots$ satisfying the Poisson equations $\Delta w = v, \cdots$ in a distributional sense.

However, the latter hierarchy of functions u, v, $\cdots$ is better displayed using some different, more systematic notation. This leads to the famous Riesz kernels, here denoted $\delta_\alpha^{(d)}$, in d-dimensional Euclidean space [29]. [These Euclidean kernels differ from the Minkowski kernels $Z_A^{(d)}$, also introduced by Riesz, and alluded to in the Introduction.] These kernels depend on a complex parameter $\alpha \in \mathbb{C}$. They are defined by

$$\delta_\alpha^{(d)}(\mathbf{x}) = K_\alpha\, r^{\alpha-d}\,, \tag{B12a}$$

$$K_\alpha = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right)}\,. \tag{B12b}$$

For any $\alpha \in \mathbb{C}$, and also for any $d \in \mathbb{C}$, the Riesz kernels satisfy the recursive relations

$$\Delta\delta_{\alpha+2}^{(d)} = -\delta_\alpha^{(d)}\,. \tag{B13}$$

Furthermore, they obey also an interesting convolution relation, which reads simply, with the chosen normalization of the coefficients K_α, as

$$\delta_\alpha^{(d)} * \delta_\beta^{(d)} = \delta_{\alpha+\beta}^{(d)}\,. \tag{B14}$$

When $\alpha = 0$ we recover the Dirac distribution in d dimensions, $\delta_0^{(d)} = K_0\, r^{-d} = \delta^{(d)}$ (the coefficient vanishes in this case, $K_0 = 0$), and we have $u = 4\pi\, \delta_2^{(d)}$, $v = -4\pi\, \delta_4^{(d)}$, $\cdots$.

The convolution relation (B14) is nothing but an elegant formulation of the Riesz formula in d dimensions. To check it let us consider the Fourier transform of r^α in d dimensions,

$$\widetilde{f}_\alpha(\mathbf{k}) \equiv \int d^d\mathbf{x}\,|\mathbf{x}|^\alpha e^{-i\mathbf{k}.\mathbf{x}}\,. \tag{B15}$$

Using (B1) we can rewrite it as

$$\widetilde{f}_\alpha(\mathbf{k}) = \int_0^{+\infty} dr r^{\alpha+d-1}\int d\Omega_{d-1}e^{-i\mathbf{k}.\mathbf{x}}\,, \tag{B16}$$

in which the angular integration can be performed as an application of Eq. (B4). This yields an expression depending on the usual Bessel function,[28]

$$\int d\Omega_{d-1}e^{-i\mathbf{k}.\mathbf{x}} = \Omega_{d-2}\int_0^\pi d\theta_{d-1}\left(\sin\theta_{d-1}\right)^{d-2}e^{-ik\,r\cos\theta_{d-1}} = (2\pi)^{\frac{d}{2}}\left(k\,r\right)^{1-\frac{d}{2}}J_{\frac{d}{2}-1}(k\,r)\,, \tag{B17}$$

where $k \equiv |\mathbf{k}|$. The radial integration in Eq. (B16) is then readily done from using the previous expression, and we obtain

$$\widetilde{f}_\alpha(\mathbf{k}) = 2^{\alpha+d}\,\pi^{\frac{d}{2}}\,\frac{\Gamma\left(\frac{\alpha+d}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)}\,k^{-\alpha-d}\,, \tag{B18}$$

where the factor in front of the power $k^{-\alpha-d}$, say A_α, is checked from the Parseval theorem for the inverse Fourier transform, which necessitates that $A_\alpha A_{-\alpha-d} = (2\pi)^d$. Finally we can check the Riesz formula by going to the Fourier domain, using the previous relations. The result,

$$\int d^d\mathbf{x}\,r_1^\alpha r_2^\beta = \pi^{\frac{d}{2}}\frac{\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\beta+d}{2}\right)\Gamma\left(-\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2d}{2}\right)}\,r_{12}^{\alpha+\beta+d}\,, \tag{B19}$$

is equivalent to Eq. (B14).

A set of formulae concerning symmetric-trace-free (STF) multipole expansions in d dimensions is presented next, without proofs. We use the multi-index notation $L = i_1\cdots i_\ell$; more generally the notation is the same as in Appendix A of [47]. In particular $\hat{n}_L$ denotes the STF projection of $n_L = n_{i_1}\cdots n_{i_\ell}$, $[\frac{\ell}{2}]$ means the integer part of $\frac{\ell}{2}$, $T_{\{i_1\cdots i_\ell\}}$ denotes the (unnormalized, minimal) sum of $T_{i_{\sigma(1)}\cdots i_{\sigma(\ell)}}$ where the σ's are permutations of the indices such that $T_{\{i_1\cdots i_\ell\}}$ is fully symmetric in L (for convenience we do not normalize the latter

[28] We adopt for the Bessel function the defining expression

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^1 dx\left(1-x^2\right)^{\nu-\frac{1}{2}}e^{-izx}\,.$$

To obtain Eq. (B18) we employ the integration formula

$$\int_0^{+\infty} dz\,z^\mu J_\nu(z) = 2^\mu\,\frac{\Gamma\left(\frac{1+\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-\mu+\nu}{2}\right)}\,.$$

sum, for instance $\delta_{\{ij}n_{k\}} = \delta_{ij}n_k + \delta_{ik}n_j + \delta_{jk}n_i$).

$$
\begin{aligned}
n_L &= \sum_{k=0}^{[\frac{\ell}{2}]} a_\ell^k \delta_{\{i_1 i_2} \cdots \delta_{i_{2k-1} i_{2k}} \hat{n}_{L-2K\}} \,, &\text{(B20a)}\\
\hat{n}_L &= \sum_{k=0}^{[\frac{\ell}{2}]} b_\ell^k \delta_{\{i_1 i_2} \cdots \delta_{i_{2k-1} i_{2k}} n_{L-2K\}} \,, &\text{(B20b)}
\end{aligned}
$$

where the coefficients are

$$
\begin{aligned}
a_\ell^k &= \frac{1}{2^k} \frac{\Gamma\left(\frac{d}{2} + \ell - 2k\right)}{\Gamma\left(\frac{d}{2} + \ell - k\right)} \,, &\text{(B21a)}\\
b_\ell^k &= \frac{(-)^k}{2^k} \frac{\Gamma\left(\frac{d}{2} + \ell - k - 1\right)}{\Gamma\left(\frac{d}{2} + \ell - 1\right)} \,. &\text{(B21b)}
\end{aligned}
$$

In particular (the brackets $\langle\,\rangle$ surrounding the indices mean the STF projection)

$$
\begin{aligned}
n_i \hat{n}_L &= \hat{n}_{iL} + \frac{\ell}{d + 2\ell - 2} \delta_{i\langle i_\ell} \hat{n}_{L-1\rangle} \,, &\text{(B22a)}\\
n_i \hat{n}_{iL} &= \frac{d + \ell - 2}{d + 2\ell - 2} \hat{n}_L \,. &\text{(B22b)}
\end{aligned}
$$

Spherical averages:

$$
\begin{aligned}
\int \frac{d\Omega_{d-1}}{\Omega_{d-1}} n_{2P} &= \frac{1}{2^p} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} + p\right)} \delta_{\{i_1 i_2} \cdots \delta_{i_{2p-1} i_{2p}\}} \,, &\text{(B23a)}\\
\hat{f}_P \hat{g}_Q \int \frac{d\Omega_{d-1}}{\Omega_{d-1}} n_{PQ} &= \delta_{p,q} \frac{p!}{2^p} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} + p\right)} \hat{f}_P \hat{g}_P \,. &\text{(B23b)}
\end{aligned}
$$

STF decomposition of a scalar function:

$$
\begin{aligned}
f(\mathbf{n}) &= \sum_{\ell=0}^{+\infty} \hat{f}_L \hat{n}_L \,, &\text{(B24a)}\\
\hat{f}_L &= \frac{2^{\ell-1} \Gamma\left(\frac{d}{2} + \ell\right)}{\ell! \pi^{\frac{d}{2}}} \int d\Omega_{d-1} \hat{n}_L f(\mathbf{n}) \,. &\text{(B24b)}
\end{aligned}
$$

Decomposition of a function $F(\mathbf{n}.\mathbf{n}')$ in terms of Gegenbauer polynomials:[29]

$$F(\mathbf{n}.\mathbf{n}') \;=\; \frac{\Gamma\left(d-2\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\sum_{\ell=0}^{+\infty}\frac{2^\ell\Gamma\left(\frac{d}{2}+\ell\right)}{\Gamma\left(d+\ell-2\right)}\hat{n}_L\hat{n}'_L\int_{-1}^{+1}dx\left(1-x^2\right)^{\frac{d-3}{2}}C_\ell^{\frac{d}{2}-1}(x)F(x)\,, \tag{B25a}$$

$$\hat{n}_L\hat{n}'_L \;=\; \frac{\ell!}{2^\ell}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}+\ell-1\right)}C_\ell^{\frac{d}{2}-1}(\mathbf{n}.\mathbf{n}')\,. \tag{B25b}$$

Integration formulae:

$$\Delta^{-1}r^\alpha \;=\; \frac{r^{\alpha+2}}{(\alpha+2)(\alpha+d)}\,, \tag{B26a}$$

$$\Delta^{-n}r^\alpha \;=\; \frac{\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma\left(\frac{\alpha+d}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+n+1\right)\Gamma\left(\frac{\alpha+d}{2}+n\right)}\frac{r^{\alpha+2n}}{2^{2n}}\,, \tag{B26b}$$

$$\Delta^{-1}\left(\hat{n}_L r^\alpha\right) \;=\; \frac{\hat{n}_L r^{\alpha+2}}{(\alpha-\ell+2)(\alpha+\ell+d)}\,. \tag{B26c}$$

APPENDIX C: EXPLICIT FORM OF $g \equiv \Delta^{-1}\left(r_1^{2-d}\,r_2^{2-d}\right)$ IN d DIMENSIONS

A very important technical fact which allowed one to compute analytically the $d = 3$ equations of motion is the possibility to obtain explicitly the quadratically non linear potentials, *i.e.*, to evaluate in closed form the integrals appearing in the PN expansion of the cubic-vertex diagram of Fig. 7.

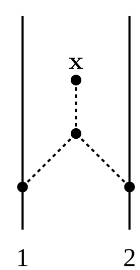

FIG. 7: Cubic-vertex diagram related to function $g \equiv \Delta^{-1}\left(r_1^{2-d}\,r_2^{2-d}\right)$.

At the lowest approximation in the c^{-1} expansion, the diagram of Fig. 7 leads, in $d = 3$, to the integral

$$g^{(d=3)}(\mathbf{x}) = \Delta^{-1}\left(\frac{1}{r_1\,r_2}\right), \tag{C1}$$

[29] By definition, the Gegenbauer polynomial $C_\ell^\gamma(x)$ is the coefficient of α^ℓ in the expansion

$$\left(1-2x\alpha+\alpha^2\right)^{-\gamma} = \sum_{\ell=0}^{+\infty}C_\ell^\gamma(x)\alpha^\ell\,.$$

The particular polynomial $P_\ell^{(d)}(x) \equiv C_\ell^{\frac{d}{2}-1}(x)$ represents an appropriate generalization of the Legendre polynomial in d dimensions [indeed $P_\ell^{(3)}(x) = P_\ell(x)$].

which was (probably) first evaluated by Fock in 1939 ("Sur le mouvement des masses finies d'après la théorie de la gravitation einsteinienne" [48]), with the simple result

$$g^{(d=3)}(\mathbf{x}) = \ln(r_1 + r_2 + r_{12}) \,. \tag{C2}$$

Remembering that $r_1 \equiv |\mathbf{x}-\mathbf{y}_1|$, $r_2 \equiv |\mathbf{x}-\mathbf{y}_2|$ and $r_{12} \equiv |\mathbf{y}_1-\mathbf{y}_2|$ the combination $r_1+r_2+r_{12}$ entering the logarithm in Eq. (C2) is simply seen to be the perimeter of the triangle joining the three spatial points $\mathbf{x}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ entering the (Newtonian approximation of the) diagram of Fig. 7. At the $\mathcal{O}(c^{-2})$ level of the PN expansion of Fig. 7, there enter several new integrals which can be reduced to

$$f^{(d=3)} = 2\,\Delta^{-1}\, g^{(d=3)} \,, \tag{C3}$$

together with

$$f_{12}^{(d=3)} = \Delta^{-1}\left(\frac{r_1}{r_2}\right) \quad \text{and} \quad 1 \leftrightarrow 2 \,. \tag{C4}$$

The explicit evaluation of the integrals (C3), (C4) is also possible, as was shown in Refs. [49],[22] (drawing on earlier works [50, 51]). In this Appendix we shall explicitly evaluate the d-dimensional generalization of (C1). It will be clear, however, that our method can be rather straightforwardly generalized to the $\mathcal{O}(c^{-2})$ diagrams contained in Fig. 7, *i.e.*, to the d-dimensional generalizations of (C3)-(C4).

For our present purpose it will be more convenient not to include the two factors of $\tilde{k}$ that accompany the two propagators issued from 1 and 2 in Fig. 7. We shall therefore define

$$g(\mathbf{x}) \equiv \Delta^{-1}(r_1^{2-d}\, r_2^{2-d}) \,. \tag{C5}$$

The method we present here consists of four basic steps: (i) expand the integrand in series and construct a corresponding series for a *particular* solution $g_{\text{part}} = \Delta^{-1}\,(r_1^{2-d}\, r_2^{2-d})_{\text{part}}$, (ii) resum the series to get an explicit line-integral form of g_{part}, (iii) compute $\Delta\, g_{\text{part}}$ in a distributional sense to discover that it satisfies $\Delta\, g_{\text{part}} = r_1^{2-d}\, r_2^{2-d} + S$ where S is a distributional source (localized along a line), and finally (iv) subtract $\Delta^{-1}S$ (which is given by another line-integral) from g_{part} to get g as a sum of line-integrals (which are expressible in terms of one special function of one argument). What is crucial in the argument is the uniqueness of the global solution (decaying at infinity) of any (distributional) Poisson equation $\Delta\,\varphi = \sigma$ when the (distributional) source decays fast enough (or, at least, does not grow too fast) at infinity. In our case, the sources σ involved will have fast-enough decay at infinity if we analytically continue d toward large enough real parts (say $\Re\,[d] > 3$).

There are several ways of implementing our method. For instance, we could start by expanding r_2^{2-d} in the source of (C5) in powers of r_1, such an expansion being valid only in a neighborhood of $\mathbf{y}_1$. Namely, we have the d-dimensional generalization of the familiar $d = 3$ Legendre-polynomial expansion of $|\mathbf{x} - \mathbf{y}_2|^{-1}$ near $\mathbf{x} = \mathbf{y}_1$ (more precisely in the ball $r_1 < r_{12}$)

$$r_2^{2-d} = r_{12}^{2-d} \sum_{\ell \geq 0} \left(\frac{r_1}{r_{12}}\right)^{\ell} P_\ell^{(d)}(c_1) \,. \tag{C6}$$

Here, we denoted for visual clarity $P_\ell^{(d)}(x) \equiv C_\ell^{\frac{d}{2}-1}(x)$, where $C_\ell^{\gamma}(x)$ is a Gegenbauer polynomial such that $C_\ell^{1/2}(x) = P_\ell^{(3)}(x)$ is the usual $d = 3$ Legendre polynomial [see also Appendix B above]. The quantity c_1 in (C6) denotes the cosine of the angle θ_1 between $\mathbf{x} - \mathbf{y}_1$ and $\mathbf{y}_2 - \mathbf{y}_1$. The notation we shall use is summarized in Fig. 8.

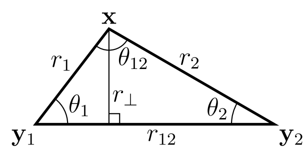

FIG. 8: Notation for various angles and distances, $\mathbf{y}_1$ and $\mathbf{y}_2$ denoting the positions of the two particles, and $\mathbf{x}$ the field point.

When inserting the *local* expansion (C6) into the source of (C5) we are led to solving (locally) an equation of the form $\Delta\, g_{\rm loc} = \sum a_\ell\, r_1^{\ell+2-d}\, P_\ell^{(d)}(c_1)$. However, using the general formula

$$\Delta(r^\lambda\, \hat{n}_L) = (\lambda-\ell)(\lambda+d-2+\ell)\, r^{\lambda-2}\, \hat{n}_L\,, \tag{C7}$$

we know a particular solution of $\Delta\,\varphi = r^\lambda\,\hat{n}_L$, namely

$$\Delta^{-1}(r^\lambda\, \hat{n}_L) = \frac{r^{\lambda+2}\,\hat{n}_L}{(\lambda+2-\ell)(\lambda+d+\ell)}\,. \tag{C8}$$

The formulae (C7)-(C8) apply to any source with fixed multipolarity (ℓ) and a power law dependence on a radius. In particular, they apply when $r \to r_1$, $\lambda \to \ell+2-d$ and $\hat{n}_L \equiv n_1^{\langle i_1 \ldots i_\ell\rangle} \to P_\ell^{(d)}(c_1)$ (because a generalized Legendre polynomial is just proportional to the contraction of an STF-projected multi unit vector $\hat{n}^L$ onto a fixed "z" direction; see Appendix B). This leads to a corresponding expansion of a *local* solution $g_{\rm loc\,1}$ (near $\mathbf{y}_1$) of $\Delta\, g_{\rm loc\,1} = r_1^{2-d}\, r_2^{2-d}$ of the form

$$g_{\rm loc\,1} = \frac{r_{12}^{2-d}\, r_1^{3-d}}{2(4-d)} \sum_{\ell\geq 0} \frac{1}{\ell+1}\, \frac{r_1^{\ell+1}}{r_{12}^{\ell}}\, P_\ell^{(d)}(c_1)\,. \tag{C9}$$

In order to proceed further, we now need to resum the expansion (C9). This is done by a trick introduced, in a similar context of resummation of multipolar expansions containing extra ℓ-dependent denominators, by Ref. [52]. One introduces some radial-integration operators $R_\alpha\,[\phi]\,(\mathbf{r}) = \int_0^1 d\lambda\, \lambda^\alpha\, \phi(\lambda\,\mathbf{r})$. For instance, in the context of (C9), one replaces $r_1^\ell/(\ell+1)$ by $R_0[r_1^\ell] = \int_0^1 d\lambda\, (\lambda\, r_1)^\ell$ or equivalently $r_1^{\ell+1}/(\ell+1)$ by $\int_0^{r_1} d\ell_1\, \ell_1^\ell$. This transforms back the multipolar series appearing in (C9) into the original "Legendre" series entering Eq. (C6). This allows one to write $g_{\rm loc\,1}$ as a simple line-integral:

$$\begin{aligned} g_{\rm loc\,1} &= \frac{r_1^{3-d}}{2(4-d)} \int_0^{r_1} d\ell_1\, |\mathbf{y}_{\ell_1} - \mathbf{y}_2|^{2-d} \\ &= \frac{r_1^{4-d}}{2(4-d)} \int_0^1 d\alpha\, |\mathbf{y}_\alpha - \mathbf{y}_2|^{2-d}\,. \end{aligned} \tag{C10}$$

Here, $\mathbf{y}_{\ell_1}$ is a point on the segment joining $\mathbf{y}_1$ to $\mathbf{x}$, located a distance ℓ_1 away from $\mathbf{y}_1$. It is more convenient to replace the line-integration over the dimensionful length ℓ_1 ($0 \leq \ell_1 \leq r_1$) into an integration over the dimensionless parameter $\alpha \equiv \ell_1/r_1$ ($0 \leq \alpha \leq 1$). This leads to the explicit expression

$$\mathbf{y}_{\ell_1} \equiv \mathbf{y}_\alpha = \alpha\,\mathbf{x} + (1-\alpha)\,\mathbf{y}_1\,. \tag{C11}$$

The resummed line-integral expression (C10) allows one to define everywhere $g_{\rm loc\,1}$, *including in the domain* $r_1 > r_{12}$ where the original series (C9) was *not* convergent. Having in hands

such a global definition of $g_{\mathrm{loc}\,1}$ then allows one to compute its Laplacian, *in the sense of distributions*, and to see how it differs from $r_1^{2-d}\, r_2^{2-d}$. The calculation of $\Delta\, g_{\mathrm{loc}\,1}$ is done by techniques similar to the ones used in Ref. [51]. One needs to rewrite some terms in the form of α-derivatives. For instance, several of the terms appearing in $\Delta\, g_{\mathrm{loc}\,1}$ can be rewritten as the line-integral

$$\begin{aligned}\int_0^1 d\alpha\, r_1^{2-d}\left[|\mathbf{y}_\alpha - \mathbf{y}_2|^{2-d} + \alpha\, \frac{\partial}{\partial\alpha}\, |\mathbf{y}_\alpha - \mathbf{y}_2|^{2-d}\right] \\ = \int_0^1 d\alpha\, \frac{\partial}{\partial\alpha}\left[\alpha\, r_1^{2-d}\, |\mathbf{y}_\alpha - \mathbf{y}_2|^{2-d}\right] = r_1^{2-d}\, r_2^{2-d}\,,\end{aligned} \tag{C12}$$

where the last line-integral gave only the end contribution $\alpha = 1$ corresponding to $\mathbf{y}_\alpha = \mathbf{x}$. Besides the terms yielding (C12), *i.e.*, the looked-for "source" of the complete g, the calculation of $\Delta\, g_{\mathrm{loc}\,1}$ yields also the distributional source (where $k \equiv \tilde{k}/4\pi$ entered through $\Delta\, r^{2-d} = -\delta^{(d)}/k$)

$$-\frac{r_1^{4-d}}{2(4-d)\,k} \int_0^1 d\alpha\, \alpha^2\, \delta^{(d)}\left(\mathbf{y}_\alpha - \mathbf{y}_2\right). \tag{C13}$$

This is conveniently transformed by introducing $\beta \equiv 1/\alpha$ (with $1 \leq \beta \leq +\infty$) and

$$\mathbf{y}_\beta \equiv (1-\beta)\, \mathbf{y}_1 + \beta\, \mathbf{y}_2\,, \tag{C14}$$

which varies along a semi-infinite line going from $\mathbf{y}_2$ to infinity along the direction $\mathbf{y}_2 - \mathbf{y}_1$, *i.e.*, *away* from $\mathbf{y}_1$. This transformation allows one to rewrite (C13) in the more transparent form

$$-\frac{r_{12}^{4-d}}{2(4-d)\,k} \int_1^{+\infty} d\beta\, \delta^{(d)}\left(\mathbf{x} - \mathbf{y}_\beta\right). \tag{C15}$$

At this stage, we recognize in (C15) a very simple source, namely a *uniform* distribution of "mass" along the half-line along which β runs. This allows one to easily compute the unique, global (decaying at infinity) solution of the Poisson equation with source (C15) and to subtract it from $g_{\mathrm{loc}\,1}$ to get the unique, global g in the form of two line-integrals:

$$g = \frac{r_1^{3-d}}{2(4-d)} \int_0^{r_1} d\ell_\alpha\, |\mathbf{y}_\alpha - \mathbf{y}_2|^{2-d} - \frac{r_{12}^{3-d}}{2(4-d)} \int_{r_{12}}^{\infty} d\ell_\beta\, |\mathbf{x} - \mathbf{y}_\beta|^{2-d}\,, \tag{C16}$$

where $d\ell_\alpha = |d\mathbf{y}_\alpha| = r_1\, d\alpha$ and $d\ell_\beta = |d\mathbf{y}_\beta| = r_{12}\, d\beta$. In other words, (C16) expresses g as, essentially, the difference between the Newtonian potentials generated by two uniform line distributions: a segment joining $\mathbf{y}_1$ to $\mathbf{x}$ and the half-line starting from $\mathbf{y}_2$ in the direction away from $\mathbf{y}_1$. It is easily seen (modulo the slight delicacy of the logarithmic divergence of the potential of a semi-infinite line when $d \to 3$, *i.e.*, the occurrence of a $1/\varepsilon$ pole; see below) that the result (C16) yields, when $d \to 3$, the well-known result (C2). [Actually, this was the way one of us (T.D.) had derived long ago for himself (C2), unaware of its derivations in the literature.]

The expression (C16) has the advantage of being explicitly regular (except at the point $\mathbf{x} = \mathbf{y}_1$) in the ball $r_1 < r_{12}$. However, it has the default of treating dissymmetrically the two points $\mathbf{y}_1$ and $\mathbf{y}_2$ (in spite of the fact that the result (C16) for g *is*, actually, symmetric under $1 \leftrightarrow 2$). One can derive an exchange-symmetric expression for g by modifying the first step of our method. Instead of expanding the source $r_1^{2-d}\, r_2^{2-d}$ in the neighborhood of

$\mathbf{x} = \mathbf{y}_1$, *i.e.*, in a series of positive powers of r_1, we can expand it in the *neighborhood of infinity*, *i.e.*, in a series of negative powers of r_1. Such an expansion is directly related to the expansions used in [51], which led to the decomposition of $g^{(d=3)}$ in two pieces denoted k and h there, where the source of h was a uniform mass distribution along the segment joining $\mathbf{y}_1$ and $\mathbf{y}_2$. Let us briefly indicate the successive steps of this new calculation of g. Instead of the "local" expansion (C6) (valid for $r_1 < r_{12}$), one expands r_2^{2-d} near infinity ($r_1 > r_{12}$) as

$$r_2^{2-d} = r_1^{2-d} \sum_{\ell \geq 0} \left(\frac{r_{12}}{r_1} \right)^{\ell} P_\ell^{(d)}(c_1) \,. \tag{C17}$$

Solving term by term $\Delta\, g_\infty = r_1^{2-d}\, r_2^{2-d}$ "near infinity" by means of (C8), and transforming away the appearing ℓ-dependent denominators by means of $\int_{r_1}^\infty d\ell_1\, \ell_1^{2-d-\ell} = r_1^{3-d-\ell}/(\ell{+}d{-}3)$, leads to the following resummation of g_∞:

$$g_\infty = -\frac{r_1^{4-d}}{2(4-d)} \int_1^\infty d\alpha\, |\mathbf{y}_\alpha - \mathbf{y}_2|^{2-d} \,. \tag{C18}$$

Here, $\mathbf{y}_\alpha$ is still defined by (C11), but the parameter α now varies in $1 \leq \alpha \leq +\infty$ so that (C18) is the potential of a semi-infinite line. Computing the distributional Laplacian of the particular solution g_∞, Eq. (C18), leads to the presence, besides the looked-for source $r_1^{2-d}\, r_2^{2-d}$, of an additional distributional source localized now along the segment joining $\mathbf{y}_1$ to $\mathbf{y}_2$, namely

$$\frac{r_{12}^{4-d}}{2(4-d)\, k} \int_0^1 d\beta\, \delta^{(d)}\left(\mathbf{x} - \mathbf{y}_\beta\right) , \tag{C19}$$

where $\beta = 1/\alpha$ varies between 0 and 1 and where $\mathbf{y}_\beta$ is again defined by (C14). It is then easy to subtract from g_∞ (which tends, in $d = 3$, to the function $k(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$ of [51]) the Poisson integral of the source (C19) (which is a uniform distribution along the segment $\mathbf{y}_1$-$\mathbf{y}_2$ and which tends, in $d = 3$, to the function $\frac{1}{2} h(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$ of [51]) to get the following alternative expression for g,

$$g = -\frac{r_1^{3-d}}{2(4-d)} \int_{r_1}^\infty d\ell_\alpha\, |\mathbf{y}_2 - \mathbf{y}_\alpha|^{2-d} + \frac{r_{12}^{3-d}}{2(4-d)} \int_0^{r_{12}} d\ell_\beta\, |\mathbf{x} - \mathbf{y}_\beta|^{2-d} \,, \tag{C20}$$

or, equivalently,

$$g = -\frac{r_1^{4-d}}{2(4-d)} \int_1^\infty d\,\alpha\, |\mathbf{y}_2 - \mathbf{y}_\alpha|^{2-d} + \frac{r_{12}^{4-d}}{2(4-d)} \int_0^1 d\,\beta\, |\mathbf{x} - \mathbf{y}_\beta|^{2-d} \,. \tag{C21}$$

The form (C20)-(C21) still does not look quite symmetric between 1 and 2 but a moment of reflection will show that it is.

The two methods above have expressed g in terms of the Newtonian potentials generated by half-lines or segments, *i.e.*, integrals of the type $\int d\ell\, |\mathbf{x}' - \mathbf{y}_\ell|^{2-d}$ where $\mathbf{y}_\ell$ varies along a straight line (but where $\mathbf{x}'$ might be $\mathbf{x}$ or $\mathbf{y}_2$). Clearly, any such potential can be reduced (through linear decompositions) to the Newtonian potential generated by a *half-line*. Let us then consider a generic half-line starting at the point $\mathbf{r}_0$ and going to infinity in the direction $\mathbf{n}$, and let us consider the Newtonian potential generated by this half-line at the origin of the coordinate system (not located on the half-line). Let us denote $\mathbf{r}_\ell = \mathbf{r}_0 + \ell\, \mathbf{n}$, $r_\ell = |\mathbf{r}_\ell|$, $\mathbf{r}_0 = r_0\, \mathbf{n}_0$, $c = \mathbf{n}_0 \cdot \mathbf{n}$ (cosine of the angle θ between the radius vector from the origin,

i.e., the "field point", towards the beginning point of the half-line and the direction of the half-line, away from its beginning). Then it is easy to find that

$$\int_0^{\infty} d\ell \, r_\ell^{2-d} = \varphi(c)\, r_0^{3-d}\,, \tag{C22}$$

where the function $\varphi(c)$ is given by the integral

$$\varphi(c) \equiv \int_0^{\infty} \frac{d\,\lambda}{(1+2\,c\,\lambda+\lambda^2)^{\frac{d-2}{2}}}\,. \tag{C23}$$

The integral (C23) converges for $d>3$, has a pole $\propto 1/(d-3)$ as $d\to 3$, and can be expressed in terms of hypergeometric functions, *e.g.* $F_{2,1}\left[\frac{1}{2}, \frac{d-2}{2}; \frac{3}{2}; z\right]$. It is, however, simpler to keep the form (C23).[30]

Finally, using the half-line potentials (C22) as building blocks one can write our result (C20) in the final, $1 \leftrightarrow 2$ symmetric, form

$$\begin{aligned} g(\mathbf{x},\mathbf{y}_1,\mathbf{y}_2) \;&=\; \frac{r_{12}^{3-d}}{2(4-d)}\left[2\,r_\perp^{3-d}\,\varphi(0) - r_1^{3-d}\,\varphi(c_1) - r_2^{3-d}\,\varphi(c_2)\right] \\ &-\; \frac{r_1^{3-d}\,r_2^{3-d}}{2(4-d)}\,\varphi(c_{12})\,. \end{aligned} \tag{C24}$$

The quantities entering (C24) are those defined in Fig. 8, notably $c_1 \equiv \cos\theta_1$, $c_2 \equiv \cos\theta_2$, $c_{12} \equiv \cos\theta_{12}$, with $r_\perp$ being the orthogonal distance between the field point $\mathbf{x}$ and the segment joining $\mathbf{y}_1$ to $\mathbf{y}_2$ [with associated argument $c_\perp = \cos\frac{\pi}{2} = 0$ in $\varphi(c)$]. Note the following properties of the function $\varphi(c)$,

$$\varphi(c)+\varphi(-c) = \frac{2\,\varphi(0)}{(1-c^2)^{\frac{d-3}{2}}}\,, \tag{C25a}$$

$$\varphi(0) = \frac{1}{2}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d-2}{2}\right)}\,, \quad \varphi(1) = \frac{1}{d-3}\,. \tag{C25b}$$

The simplest way to prove (C25a) is to notice that the Newtonian potential of an *infinite* line can be written either as twice that of two half-lines beginning at the orthogonal projection of the field point on the original line, so that

$$\int_{-\infty}^{+\infty} \frac{d\ell}{r_\ell^{d-2}} = 2\,\frac{\varphi(0)}{r_\perp^{d-3}}\,, \tag{C26}$$

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[30] The multipolar expansion of the function $\varphi(c)$ reads

$$\varphi(c) = \sum_{\ell\geq 0} (-)^\ell \frac{2\ell+d-2}{(\ell+1)(\ell+d-3)} P_\ell^{(d)}(c)\,.$$

On this expression one sees clearly the occurrence of the simple pole of $\varphi(c)$ when $\varepsilon \equiv d-3 \to 0$, which is given by the "monopolar" term $\ell = 0$ as

$$\varphi(c) = \frac{P_0^{(3)}(c)}{\varepsilon} + \mathcal{O}(\varepsilon^0) = \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)\,.$$

or as that of two other half-lines obtained by a more arbitrary cut (under an angle $\theta \neq \pi/2$ and $c = \cos\theta$).

We can verify the $d \to 3$ limit of Eqs. (C16) and (C24) by using the following $\varepsilon \to 0$ expansion of the elementary function $\varphi(c)$, namely

$$\varphi(c) = \frac{1}{\varepsilon} - \ln\left(\frac{1+c}{2}\right) + \mathcal{O}\left(\varepsilon\right) . \tag{C27}$$

To obtain (C27) we notice that the finite part of $\varphi(c)$ when $\varepsilon \to 0$, which is

$$\varphi_0(c) \equiv \lim_{\varepsilon\to 0}\left[\varphi(c) - \frac{1}{\varepsilon}\right] , \tag{C28}$$

can be re-expressed in the form of the following sum of two convergent integrals,

$$\begin{aligned} \varphi_0(c) &= \int_0^1 \frac{d\,\lambda}{\sqrt{1+2\,c\,\lambda+\lambda^2}} + \int_0^1 \frac{d\,\lambda}{\lambda}\left(\frac{1}{\sqrt{1+2\,c\,\lambda+\lambda^2}} - 1\right) \\ &= -\ln\left(\frac{1+c}{2}\right) . \end{aligned} \tag{C29}$$

Combining the expansion (C27) with the basic relations[31] associated with the triangle of Fig. 8, our d-dimensional expressions (C16) or (C24) are found to admit the expansion

$$g = -\frac{1}{2\varepsilon} - \frac{1}{2} + \ln\left(\frac{r_1+r_2+r_{12}}{2}\right) + \mathcal{O}\left(\varepsilon\right) , \tag{C30}$$

which indeed reduces to the three-dimensional result (C2) modulo an additive constant linked to the $1/\varepsilon$ pole.

Nice as it is to have in hand an analytic expression for the d-dimensional basic non linear potential g, its practical utility in explicit computations of the d-dimensional equations of motion is not evident because, contrary to the 3-dimensional expression (C2), the expression is not *explicitly* regular along the $\mathbf{y}_1 - \mathbf{y}_2$ segment. [The regularity of Eq. (C24) as $r_\perp \to 0$ comes by compensations between the three terms in the bracket, using (C25a).] It would need some transforming [using (C25a), and/or using the other expressions derived from the previous form (C16), which are regular along the $\mathbf{y}_1 - \mathbf{y}_2$ segment, but singular somewhere else] to write an explicit expression which is regular everywhere, except at the two isolated points $\mathbf{y}_1$ and $\mathbf{y}_2$.

Finally, let us just mention that the method explained above can, in principle, be straightforwardly generalized to the computation of the higher post-Newtonian potentials contained

[31] Denoting $S \equiv r_1 + r_2 + r_{12}$ we have

$$\frac{1+c_1}{S} = \frac{r_1+r_{12}-r_2}{2r_1r_{12}} , \qquad \frac{1+c_2}{S} = \frac{r_2+r_{12}-r_1}{2r_2r_{12}} , \qquad \frac{1+c_{12}}{S} = \frac{r_1+r_2-r_{12}}{2r_1r_2} .$$

The perpendicular distance $r_\perp$ is given by

$$r_\perp = \frac{r_1r_2}{S}\sqrt{2(1+c_1)(1+c_2)(1+c_{12})} .$$

in the diagram of Fig. 7. For instance, in computing the d-dimensional analog of (C3), say $f = 2\,\Delta^{-1} g = 2\,\Delta^{-2}(r_1^{2-d}\,r_2^{2-d})$, it is easy [by iterating (C8)] to get the analog of (C9). Then a more complicated radial-integration operator (see, *e.g.*, [52]) will allow one to resum the series to get a line-integral expression for $f_{\mathrm{loc}\,1}$ or f_∞. We anticipate that a somewhat more delicate application of either Δ (to go back to g) or Δ^2 (to go back to $2\,r_1^{2-d}\,r_2^{2-d}$) will yield additional line-distributed sources. It should then be a simple matter to compute the Poisson, or iterated Poisson, integral of these line-distributed sources. We leave an explicit study of these details to future work.

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